VIBRATION ANALYSIS OF A BEAM WITH UNCERTAIN-BUT-BOUNDED PARAMETERS USING INTERVAL FINITE ELEMENT METHOD

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AKANKSHA

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DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "**Vibration analysis of a beam with uncertain-but-bounded parameters using interval finite element method"** in partial fulfillment of the requirement for the award of the degree of Master of Science, submitted in the Department of Mathematics, National Institute of Technology, Rourkela is an authentic record of my own work carried out under the supervision of Dr. S. Chakraverty.

The matter embodied in this has not been submitted by me for the award of any other degree.

(AKANKSHA) Date:

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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ABSTRACT

This thesis investigates the vibration of beam for computing its natural frequency with uncertain-but-bounded parameters i.e. interval material properties in the finite element method. The problem is formulated first using the energy equation by converting the problem to a generalized eigenvalue problem. The generalized eigenvalue problem obtained contains the mass and stiffness matrix. In general these matrices contain the crisp values of the parameters and then it is easy to solve by various well known methods. But, in actual practice there are incomplete information about the variables being a result of errors in measurements, observations, applying different operating conditions or it may be maintenance induced error, etc. Rather than the particular value of the material properties we may have only the bounds of the values. These bounds may be given in term of interval. Thus we will have the finite element equations having the interval stiffness and mass matrices. So, in turn one has to solve by the problem by interval generalized eigenvalue problem. This requires the complex interval arithmetic and so detail study of interval computation related to the present problem has been done. First homogeneous beam with crisp values of material properties are considered. Then the problem has been undertaken taking the material properties as interval. Initially, Young's modulus and density have been considered as interval separately, and then the problem has been analyzed using both Young's modulus and density properties as interval. Next, similar investigations for non-homogeneous beam have also been done. Although the non-homogeneity makes the problem more complex but this may be the actual representation of a general beam. The considered interval material properties are in term of β, where β is called the uncertainty factor. Using interval computation the interval generalized eigenvalue problem has been solved by a new proposed method. Solution of the interval eigenvalue problem gives the interval eigenvalues which are the natural frequencies in each cases of the beam as above. The computed results are shown in terms of table and plots.

The finite element method is a numerical procedure for finding approximate solutions of ordinary and partial differential equations. The solution approach is based either on elimination of the differential equation completely or rendering the differential equations into an approximate system of ordinary differential equations which are then numerically integrated using standard techniques. Finite Element method can be applied to structures, biomechanics, and fluid mechanics, electromagnetic and to many other problems. Simple linear static problems and highly complex linear and nonlinear transient dynamic problems are effectively solved using the finite element method.

Finite Element Method is being extensively used to find approximate results of complicated structures of which exact solutions cannot be found. The finite element method for the vibration problem is a method of finding approximate solutions of the governing ordinary and partial differential equations by transforming it into an eigenvalue problem.

For various scientific and engineering problems, it is an important issue how to deal with variables and parameters of uncertain value. Generally, the parameters are taken as constant for simplifying the problem. But, actually there are incomplete information about the variables being a result of errors in measurements, observations, applying different operating conditions or it may be maintenance induced error, etc. Rather than the particular value of the material properties we may have only the bounds of the values. Recently investigations are carried out by various researchers throughout the globe by taking the uncertainty in term of interval in the material properties.

This thesis investigates the vibration of uncertain beam viz, with interval material properties using finite element method. The problem has been analyzed taking Young's modulus and density properties as interval. Governing vibration equation with interval material is solved. As mentioned that a generalized interval eigenvalue problem is finally obtained when the finite element method is used in the vibration of beam with interval material properties. Solution of the corresponding interval eigenvalue problem gives the interval eigenvalues/vibration characteristics. As such obtained solutions viz. the interval eigenvalues are shown in term of interval plots. Comparison has also been made in special cases.

As mentioned above the finite element method is applied with the interval material properties hence interval computations are required for the present formulations. As such interval arithmetic is presented next for the sake of completeness.

1.1 Interval Arithmetic

Interval: An interval A is a subset of R such that $A=[a_1,a_2]=\{t \mid a_1 \le t \le a_2, a_1, a_2 \in R\}$.

If $A=[a_1,a_2]$ and $B=[b_1,b_2]$ are two intervals, then the arithmetic operations are:

Addition: $A+B=[a_1+b_1, a_2+b_2]$

Subtraction: $A-B=[a_1-b_2, a_2-b_1]$

Product: $A \bullet B=[\min\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}, \max\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}]$

Division: $A/B=[\min(a_1/b_1, a_1/b_2, a_2/b_1, a_2/b_2), \max(a_1/b_1, a_1/b_2, a_2/b_1, a_2/b_2)]$

Where , $b_1, b_2 \neq 0$.

Chapter 2 . Literature Review and Aim

2.1 Literature Review

Recently investigations are carried out by various researchers throughout the globe by using the uncertain and interval material properties. Various generalized model of uncertainty have been applied to finite element analysis to solve the vibration and static problems by using interval parameters. Although FEM in vibration problem is well known and there exist large number of papers related to this. As such few papers that are related to interval FEM are discussed here. Dimarogonas [1] studied the interval analysis of vibrating systems, where the author presented the theory for vibrating system taking interval rotator dynamics. Ye and chen[2] proposed a moving finite element method to perform the dynamic analysis of a simply supported beam for a moving mass. Moens and Michael hanss [3] gave a general overview of recent research activities on non-probabilistic finite element analysis and its application for the representation of parametric uncertainty in applied mechanics. The overview focuses on interval as well as fuzzy uncertainty treatment in finite element analysis. Since the interval finite element problem forms the core of a fuzzy analysis, the paper first discusses the problem of finding output ranges of classical deterministic finite element problems where uncertain physical parameters are described by interval quantities. Gersem et al. [4] investigated the interval and fuzzy finite element method for obtaining the eigenvalue and frequency response function analysis of structures with uncertain parameters. Recently Nisha and S.Chakraverty [5] have studied fuzzy finite element method for a bar.

2.2 Aim

The aim of the present thesis is to first understand the traditional finite element method. In the present thesis beam structure has been considered to describe the finite element method. As already mentioned, generally, the values of variables or properties are taken as crisp but in actual case the accurate crisp values cannot be obtained. To overcome the vagueness we use interval in place of crisp values. So, next aim is to study in detail the Interval Finite Element Method (IFEM). The IFEM has been used here to study the problem of vibration of beam with uncertain-but-bounded parameters. Finally new method has been proposed here to solve interval eigenvalue problem. As such simulation with various numbers of elements with crisp and interval material properties in the vibration of a beam has been investigated here.

Chapter 3 . Structural finite element model for a beam

In this method, the given structure is divided into several elements and a suitable solution within each element is assumed. From this equations are formulated and approximate solution is obtained. Fig.1 shows a beam which may be divided into finite number of elements. To understand the methodology we divide the beam into two elements.

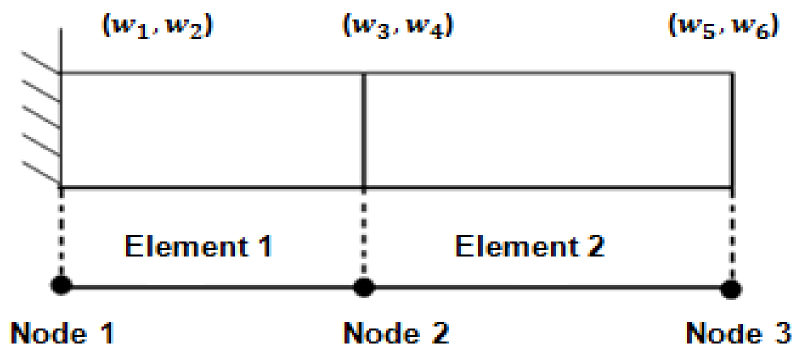

Fig. 1 Homogeneous beam discretized into two finite elements corresponding to three nodes

Let us now consider a typical i^{th} beam element as shown in Fig.2 (S.S.Rao[11]), where $l^{(i)}$, $E^{(i)}, A^{(i)}, \rho^{(i)}$ are the length , moment of inertia , area of cross-section and density of i^{th} element of the beam respectively and $w_1^{(i)}$ to $w_4^{(i)}$ denote the displacements at the ends of the element.

If x is the local co-ordinate over the beam element, the finite element approximation for the displacement must satisfy

$$u(0,t) = w_1^{(i)} (\text{vertical displacement at node } 1)$$

$$\frac{\partial u}{\partial x}(0,t) = w_2^{(i)} (\text{slop at node } 1)$$

$$u(l,t) = w_3^{(i)} (\text{vertical displacement at node } 2)$$

$$\frac{\partial u}{\partial x}(l,t) = w_4^{(i)} (\text{slop at node } 2)$$

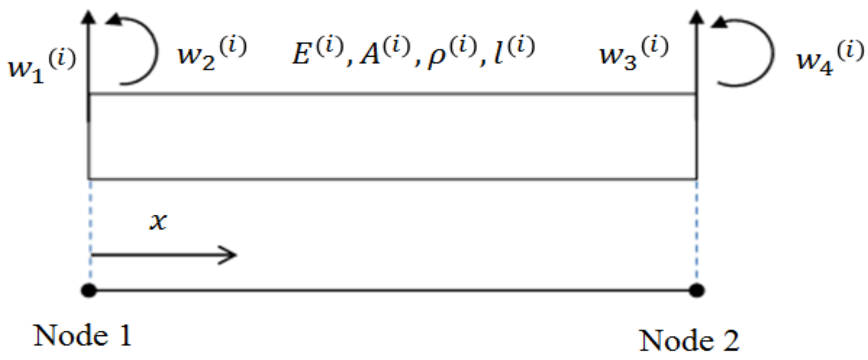

Fig.2 A typical beam element corresponding to i^{th} element

The deflection of a beam element without transverse loading across its span, but with prescribed displacements and slopes at its ends is assumed by

$$u(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3 \tag{1}$$

Using the nodal displacement conditions given above, one can obtain the values of c_1, c_2, c_3 and c_4 and then substituting these values Eq. (1), we obtain

$$\begin{aligned} u(x,t) = &\left(1 - \frac{3x^2}{\left(l^{(i)}\right)^2} + \frac{2x^3}{\left(l^{(i)}\right)^3}\right) w_1^{(i)} + \left(x - \frac{2x^2}{l^{(i)}} + \frac{x^3}{\left(l^{(i)}\right)^2}\right) w_2^{(i)} + \\ &\left(\frac{3x^2}{\left(l^{(i)}\right)^2} + \frac{2x^3}{\left(l^{(i)}\right)^3}\right) w_3^{(i)} + \left(\frac{-x^2}{l^{(i)}} + \frac{x^3}{\left(l^{(i)}\right)^2}\right) w_4^{(i)} \end{aligned} \tag{2}$$

Now the expressions for kinetic and potential energies of beam element respectively may be given as

$$K.E = \frac{1}{2}\int_0^l \rho^{(i)} A^{(i)} \left(\frac{\partial w}{\partial t}\right)^2 dx = \frac{1}{2} \vec{W}^{(i)T} \left[M^{(i)}\right] \vec{W}^{(i)}$$

$$P.E = \frac{1}{2}\int_0^l E^{(i)} I^{(i)} \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx = \frac{1}{2} \vec{W}^{(i)T} \left[K^{(i)}\right] \vec{W}^{(i)}$$

where, $E^{(i)}, I^{(i)}, \rho^{(i)}$ and $l^{(i)}$ are the components of the i^{th} element. Using Lagrange's equation one may obtain (by substituting the value of u in Eqn. (2))

$$\left[M^{(i)}\right]\left\{\ddot{U}\right\} + \left[K^{(i)}\right]\left\{U\right\} = \left\{0\right\}$$

where

$$M^{(i)} = \frac{\rho^{(i)} A^{(i)} l^{(i)}}{420} \begin{bmatrix} 156 & 22l^{(i)} & 54 & -13l^{(i)} \\ 22l^{(i)} & 4\left(l^{(i)}\right)^2 & 13l^{(i)} & -3\left(l^{(i)}\right)^2 \\ 54 & 13l^{(i)} & 156 & -22l^{(i)} \\ -13l^{(i)} & -3\left(l^{(i)}\right)^2 & -22l^{(i)} & 4\left(l^{(i)}\right)^2 \end{bmatrix}$$

and

$$K^{(i)} = \frac{E^{(i)} A^{(i)}}{\left(l^{(i)}\right)^3} \begin{bmatrix} 12 & 6l^{(i)} & -12 & 6l^{(i)} \\ 6l^{(i)} & 4\left(l^{(i)}\right)^2 & -6l^{(i)} & 2\left(l^{(i)}\right)^2 \\ -12 & -6l^{(i)} & 12 & -6l^{(i)} \\ 6l^{(i)} & 2\left(l^{(i)}\right)^2 & -6l^{(i)} & 4\left(l^{(i)}\right)^2 \end{bmatrix}$$

are the element mass and the element stiffness matrices of the i^{th} element.

In the above equation taking $U = We^{i\omega t}$ we have

$$\left[K^{(i)}\right]\left\{W\right\} = \lambda\left[M^{(i)}\right]\left\{W\right\} \tag{3}$$

where $W = \left[w_1^{(i)}, w_2^{(i)}, w_3^{(i)}, w_4^{(i)}\right]^T$ and $\lambda = \omega^2$.

Eq. (3) is a crisp generalized eigenvalue problem. Now as discussed above the material properties may not be crisp. So using finite element method for interval parameters one can obtain the interval generalized eigenvalue problem. The obtained uncertain eigenvalue problems for different material properties for different cases are discussed in the following sections.

Chapter 4 . Finite element model for homogeneous beam with crisp material properties

A fixed free beam having crisp values of material properties is considered first for determining the natural frequency. The beam is analyzed numerically with finite element models taking one and two element. For each element mass and stiffness matrices are written and then these are assembled satisfying the boundary condition. Natural frequencies are obtained after getting the global mass and stiffness matrices through assembling. The eigenvalue equations for various elements according to the boundary conditions may easily be written.

Equation for one element:

$$\frac{EI}{l^3}\begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix}\{W\} = \lambda \frac{\rho Al}{420}\begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix}\{W\} \tag{4}$$

Equation for two elements:

$$\frac{EI}{l^3}\begin{bmatrix} 24 & 0 & -12 & 6l \\ 0 & 8l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\{W\} = \lambda \frac{\rho Al}{420}\begin{bmatrix} 312 & 0 & 54 & -13l \\ 0 & 8l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}\{W\} \tag{5}$$

Let us take the values of the parameter as crisp.

$E = 2\times 10^{11} N/m^2$, $\rho = 7800 Kg/m^3$, $A = 30cm^2$, $I = 100mm^4$ and $l = 1m.$

(P.Sesu [12]).

Using these parameters along with Eq. (4) and (5), the obtained natural frequencies are given in table 1.

Table 1 Crisp value of frequencies with E, ρ, A and l as crisp

	No of elements		
	1		2
modes	1	10.6668309	10.57634112
	2	1035.48701	422.0460921
	3		4827.852251
	4		40670.25449

In the subsequent sections imprecisely defined beam viz. taking the material properties in terms of interval for homogenous cases are discussed.

Here interval values of the material properties are considered. From Eq. (3) we get the eigenvalue problem for interval values as

$$\left[\underline{K},\overline{K}\right]\{W\}=[\underline{\omega},\overline{\omega}]^2\left[\underline{M},\overline{M}\right]\{W\} \tag{6}$$

Where $\underline{K}$, $\overline{K}$, $\underline{M}$ and $\overline{M}$ are the lower and upper bounds of stiffness and mass matrices respectively,

And $\underline{\omega}$ and $\overline{\omega}$ are the lower and upper bounds of natural frequencies of the beam.

From Eq. (6), we can write the above interval eigenvalue problem as four combinations of crisp eigenvalue problem as below

$$\left[\underline{K}\right]\{W\}=\underline{\omega}_1{}^2\left[\overline{M}\right]\{W\}\ ,\ \left[\overline{K}\right]\{W\}=\overline{\omega}_1{}^2\left[\underline{M}\right]\{W\},$$

$$\left[\underline{K}\right]\{W\}=\overline{\omega}_2{}^2\left[\underline{M}\right]\{W\}\text{and }\left[\overline{K}\right]\{W\}=[\underline{\omega}_2]^2\left[\overline{M}\right]\{W\} \tag{7}$$

Here $\underline{\omega}_1<\underline{\omega}_2<\overline{\omega}_2<\overline{\omega}_1$. Now taking $\underline{\omega}=\max\,(\underline{\omega}_1,\underline{\omega}_2)$ and $\overline{\omega}=\min\,(\overline{\omega}_1,\overline{\omega}_2)$ we get the interval solution of the eigenvalue problem as $[\underline{\omega},\overline{\omega}]$.

5.1 Homogenous fixed free beam with Young's modulus as an interval

Taking $\left[\underline{E},\overline{E}\right]$, the governing equations for one and two elements according to the same boundary condition are computed. One and two element equations are incorporated here.

Equation for one element:

$$\frac{\underline{E}I}{l^3}\begin{bmatrix}12 & -6l\\ -6l & 4l^2\end{bmatrix}\{W\}=\underline{\omega}^2\,\frac{\rho Al}{420}\begin{bmatrix}156 & -22l\\ -22l & 4l^2\end{bmatrix}\{W\}\text{ and} \tag{8}$$

$$\frac{\overline{E}I}{l^3}\begin{bmatrix}12 & -6l\\ -6l & 4l^2\end{bmatrix}\{W\}=\overline{\omega}^2\,\frac{\rho Al}{420}\begin{bmatrix}156 & -22l\\ -22l & 4l^2\end{bmatrix}\{W\} \tag{9}$$

Equation for two elements:

$$\frac{\underline{EI}}{l^3}\begin{bmatrix} 24 & 0 & -12 & 6l \\ 0 & 8l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\{W\} = \underline{\omega}^2 \frac{\rho Al}{420}\begin{bmatrix} 312 & 0 & 54 & -13l \\ 0 & 8l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}\{W\} \text{ and} \tag{10}$$

(9)

$$\frac{\overline{E}I}{l^3}\begin{bmatrix} 24 & 0 & -12 & 6l \\ 0 & 8l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\{W\} = \overline{\omega}^2 \frac{\rho Al}{420}\begin{bmatrix} 312 & 0 & 54 & -13l \\ 0 & 8l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}\{W\} \tag{11}$$

Let us take the value of Young's modulus as interval.

$E = [1.998\times10^{11}, 2.002\times10^{11}]N/m^2, \rho = 7800Kg/m^3,\ A = 30cm^2, I = 100mm^4, l = 1m$.

Using these parameters along with Eq. (8),(9),(10) and (11), the obtained natural frequencies are given in table 2.

Table 2 Interval values of frequencies with Young's modulus as interval

	No of elements		
	1		2
modes	1	[10.658971,10.6802363]	[10.56549416,10.58664630]
	2	[1034.716840,1036.788345]	[421.5204515,422.3643363]
	3		[4818.600873,4828.247722]
	4		[40255.70686,40336.29887]

Taking E in terms of β, i.e. $E = [2\times10^{11} - \beta\times2\times10^{11}, 2\times10^{11} + \beta\times2\times10^{11}]$ and all other parameters are same. Where β varies from 0 to .01. Using this interval eigenvalues are obtained for the beam structure and the results obtained are depicted in term of plots which is given in fig.1 and fig.2 for 1 element discretization and in fig.3 to 6 for 2 element discretization.

For 1 element discretization:

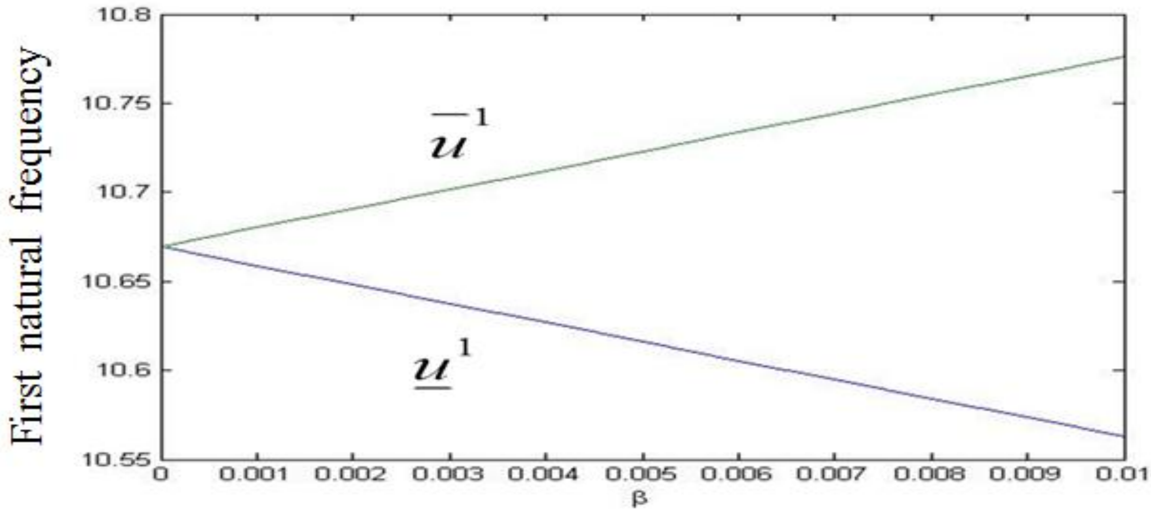

Fig.1 Plot of upper and lower bounds of first natural frequency verses the uncertainty factor β for E as interval for homogeneous beam having 1 element.

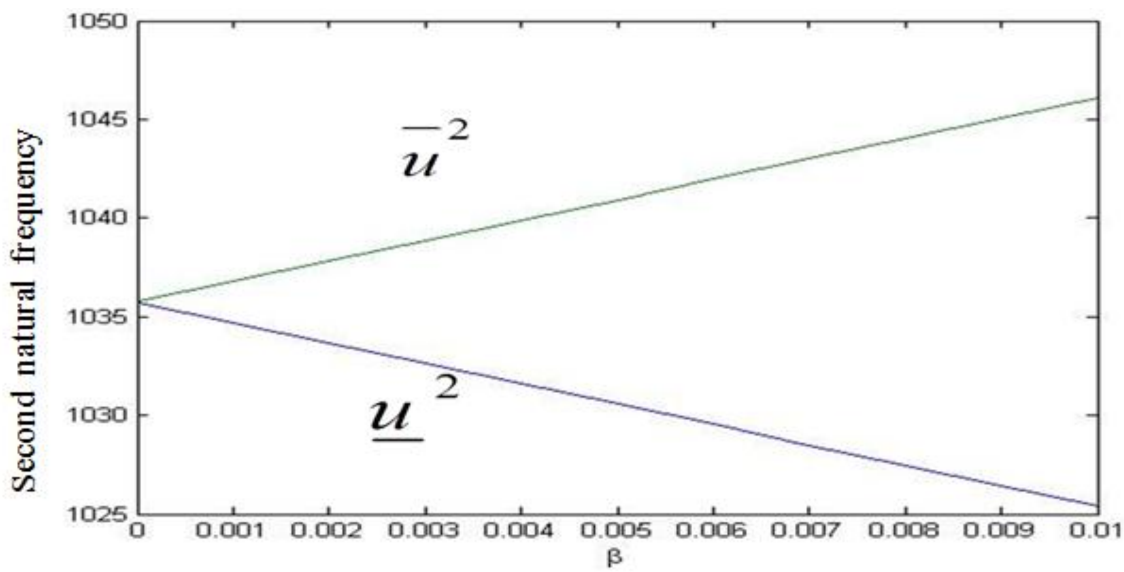

Fig.2 Plot of upper and lower bounds of second natural frequency verses the uncertainty factor β for E as interval for homogeneous beam having 1 element.

Table 3 Interval static responses of a beam having 1 element with uncertain factor $\beta = 1\%$

u	$\underline{u}^i$	$\overline{u}^i$
u^1	10.562871	10.776362
u^2	1025.3951	1046.1101

For 2 element discretization:

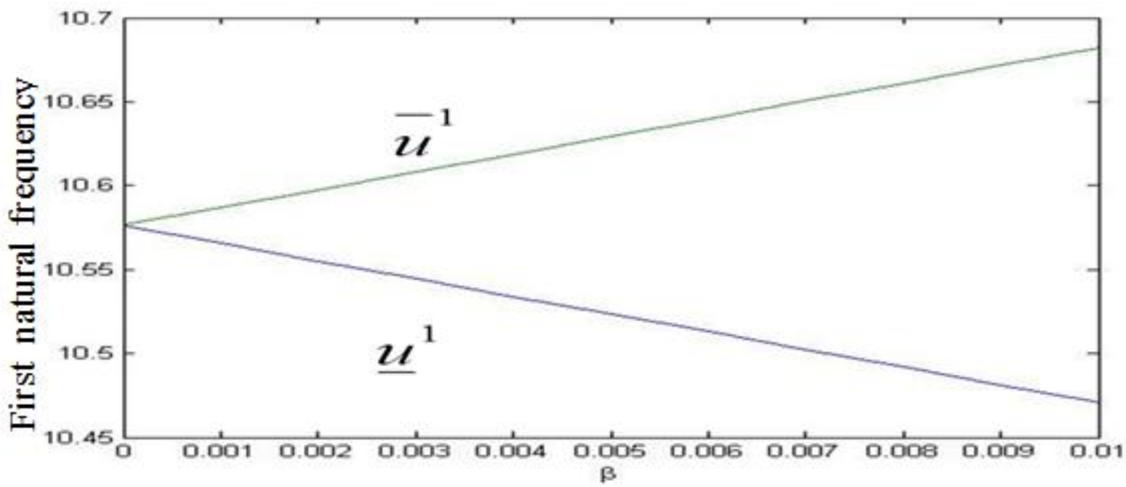

Fig.3 Plot of upper and lower bounds of first natural frequency verses the uncertainty factor β for E as interval for homogeneous beam having 2 elements.

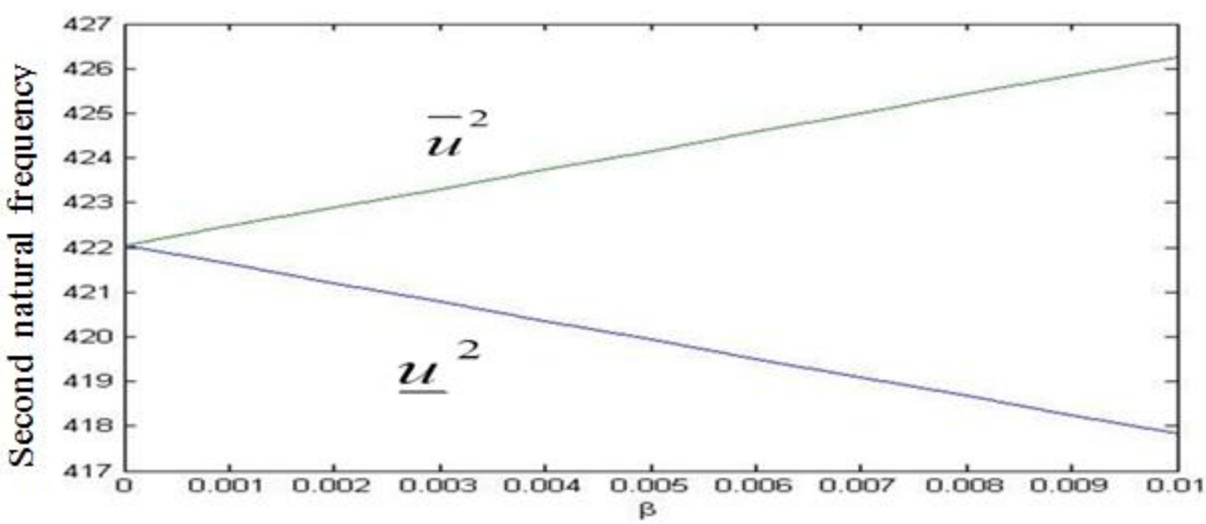

Fig.4 Plot of upper and lower bounds of second natural frequency verses the uncertainty factor β for E as interval for homogeneous beam having 2 elements.

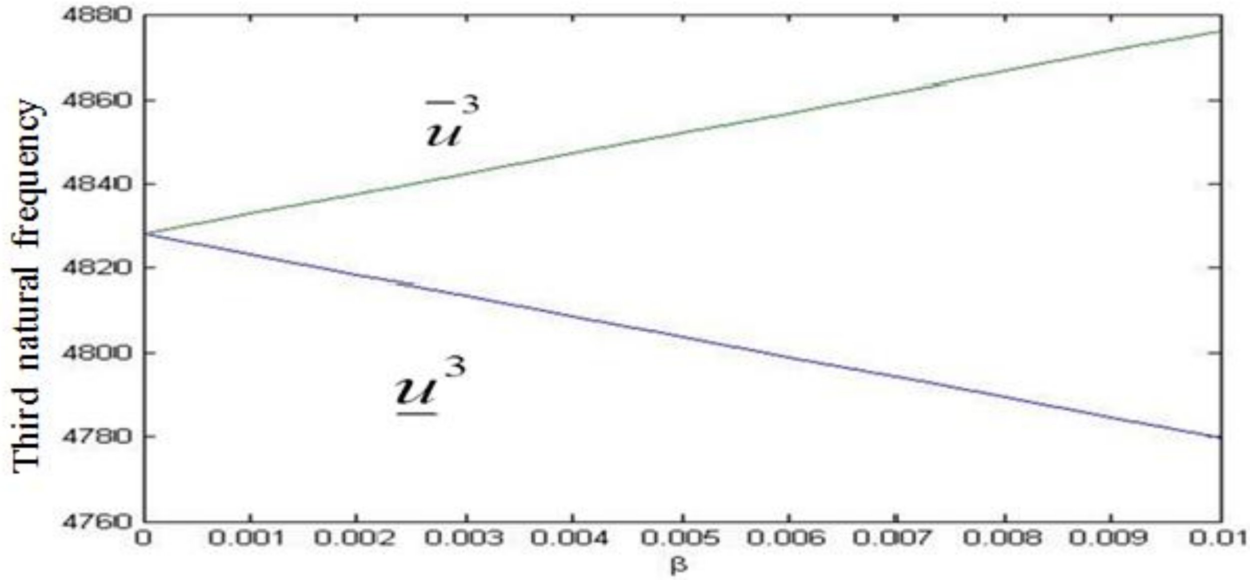

Fig.5 Plot of upper and lower bounds of third natural frequency verses the uncertainty factor β for E as interval for homogeneous beam having 2 elements.

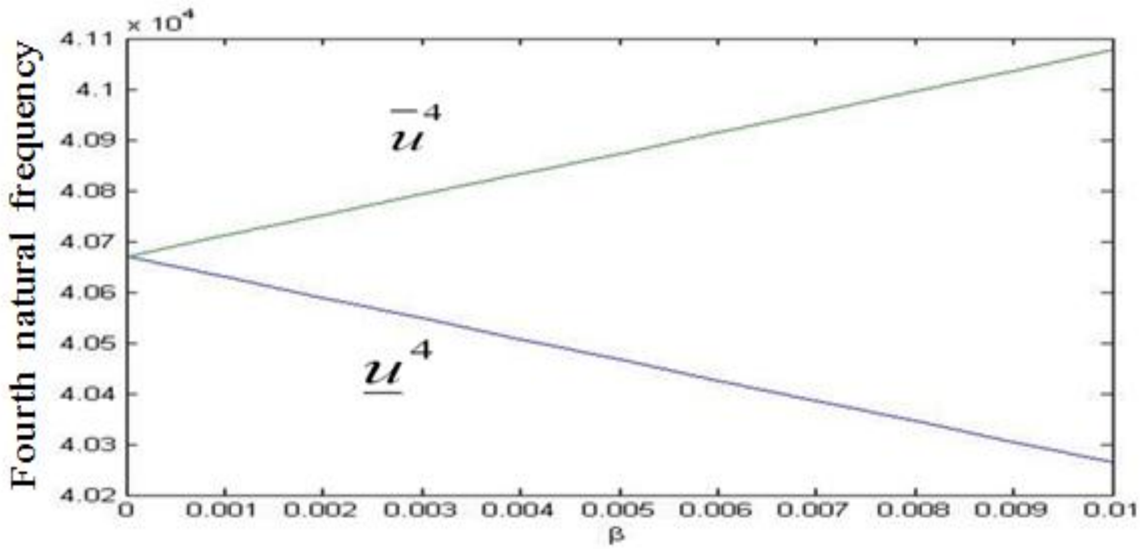

Fig.6 Plot of upper and lower bounds of fourth natural frequency verses the uncertainty factor β for E as interval for homogeneous beam having 2 elements.

Table 4 Interval static responses of a beam having 2 elements with uncertain factor $\beta = 1\%$

u	$\underline{u}^i$	$\overline{u}^i$
u^1	10.470578	10.682105
u^2	417.82563	426.26655
u^3	4779.5737	4876.1308
u^4	40263.552	41076.957

5.2 Homogenous fixed free beam with density as an interval

A homogenous fixed free beam with density as interval is considered now. The governing eigenvalue equations satisfying the boundary condition for various elements are obtained. One and two element equations are incorporated here.

Equation for one element:

$$\frac{EI}{l^3}\begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix}\{W\} = \underline{\omega}^2 \frac{\underline{\rho}Al}{420}\begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix}\{W\} \quad \text{and} \tag{12}$$

$$\frac{EI}{l^3}\begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix}\{W\} = \overline{\omega}^2 \frac{\overline{\rho}Al}{420}\begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix}\{W\} \tag{13}$$

Equation for two elements:

$$\frac{EI}{l^3}\begin{bmatrix} 24 & 0 & -12 & 6l \\ 0 & 8l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\{W\} = \underline{\omega}^2 \frac{\underline{\rho}Al}{420}\begin{bmatrix} 312 & 0 & 54 & -13l \\ 0 & 8l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}\{W\} \quad \text{and} \tag{14}$$

$$\frac{EI}{l^3}\begin{bmatrix} 24 & 0 & -12 & 6l \\ 0 & 8l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\{W\} = \overline{\omega}^2 \frac{\overline{\rho}Al}{420}\begin{bmatrix} 312 & 0 & 54 & -13l \\ 0 & 8l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}\{W\} \tag{15}$$

Let us take the density as interval.

$\rho = [7795,7805] Kg/m^3$, $E = 2\times 10^{11} N/m^2$, $A = 30cm^2, I = 100mm^4, l = 1m$.

Using these parameters along with Eq. (12),(13),(14) and (15), the obtained natural frequencies are given in table 5.

Table 5 Interval values of frequencies with density as interval

	No of elements		
	1		2
modes	1	[10.65999757,10.67367300]	[10.56009482,10.59808078]
	2	[1034.823667,1036.151215]	[421.3977878,422.9136072]
	3		[4820.436196,4837.775894]]
	4		[40607.78098,40753.85214]

Taking ρ in terms of β, i.e. $\rho = [7800 - 7800 * \beta, 7800 + 7800 * \beta]$ and all other parameters are same. . Where β varies from 0 to .01. Using these parameters interval eigenvalues are are depicted in term of plots which is given in fig.7 and fig.8 for 1 element discretization and in fig.9 to 12 for 2 element discretization.

For 1 element discretization

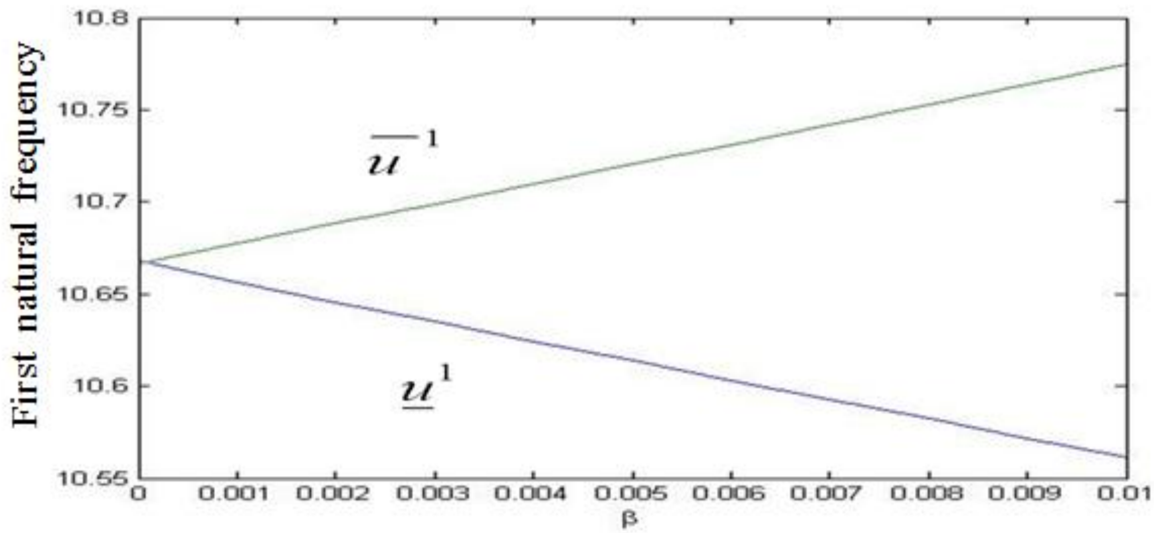

Fig.7 Plot of upper and lower bounds of first natural frequency verses the uncertainty factor β for ρ as interval for homogeneous beam having 1 element.

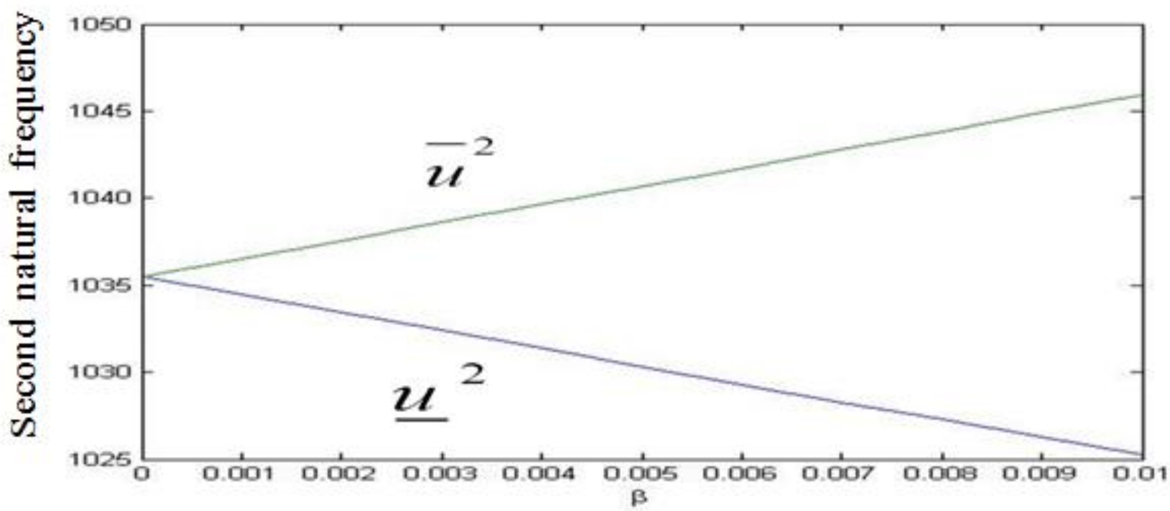

Fig.8 Plot of upper and lower bounds of second natural frequency verses the uncertainty factor β for ρ as interval for homogeneous beam having 1 element.

Table 6 Interval static responses of a beam having 1 element with uncertain factor $\beta = 1\%$

u	$\underline{u}^i$	\ $\overline{u}^i$
u^1	10.66831	10.774577
u^2	1025.2347	1045.9465

For 2 elements discretization:

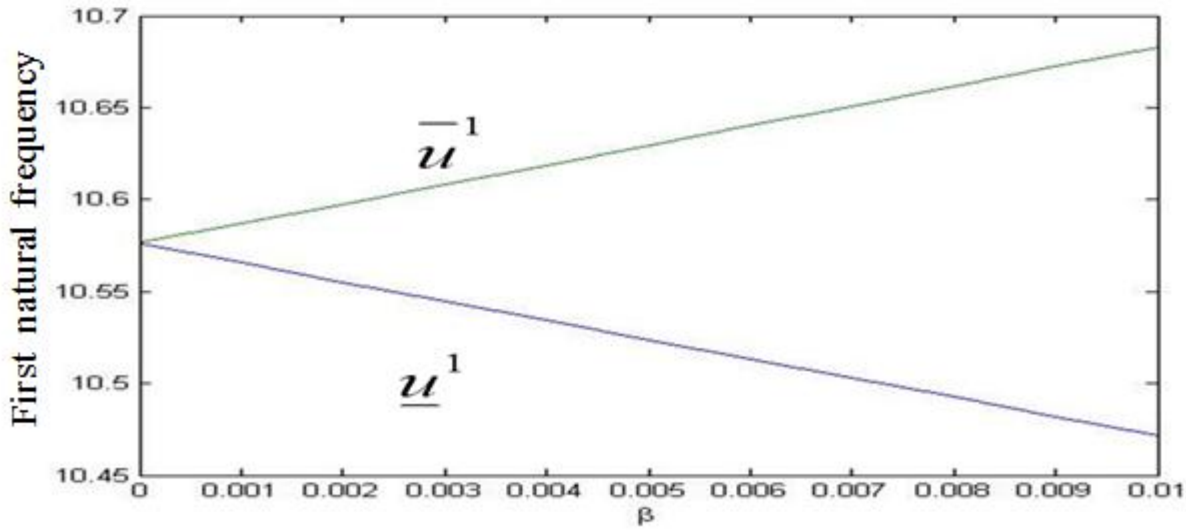

Fig.9 Plot of upper and lower bounds of first natural frequency verses the uncertainty factor β for ρ as interval for homogeneous beam having 2 elements.

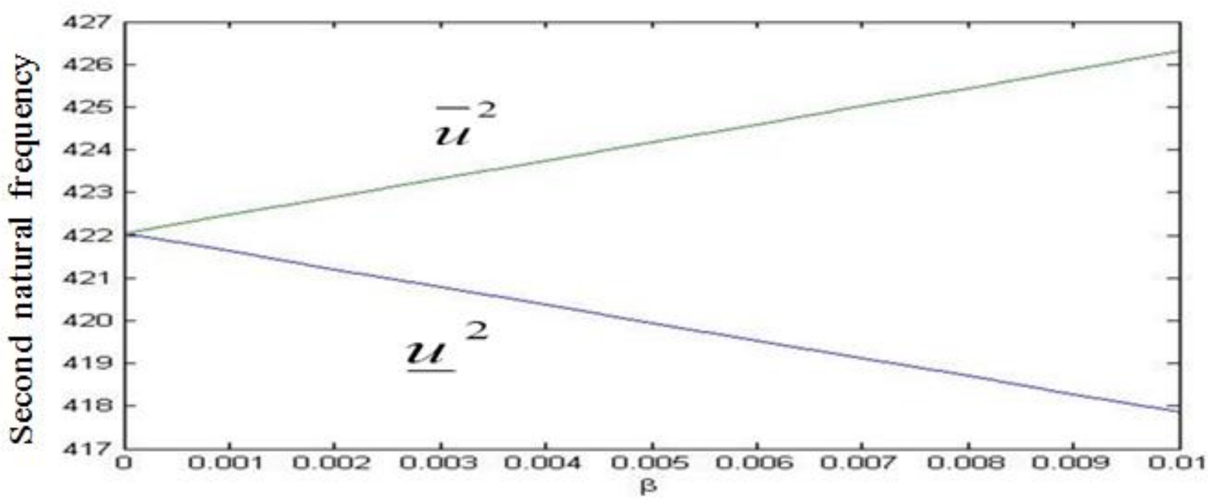

Fig.10 Plot of upper and lower bounds of second natural frequency verses the uncertainty factor β for ρ as interval for homogeneous beam having 2 elements.

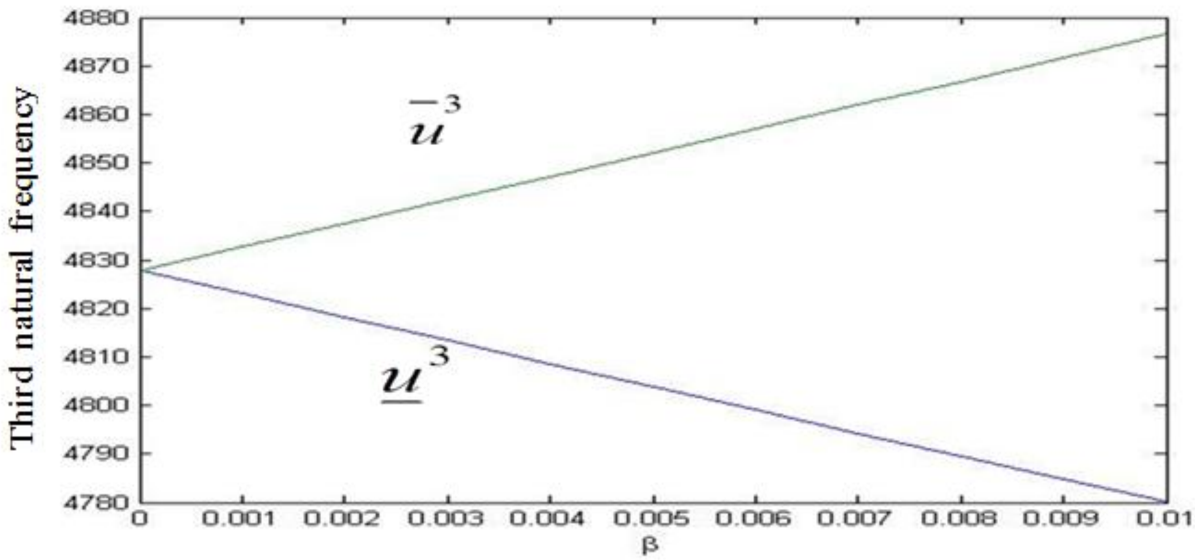

Fig.11 Plot of upper and lower bounds of third natural frequency verses the uncertainty factor β for ρ as interval for homogeneous beam having 2 elements.

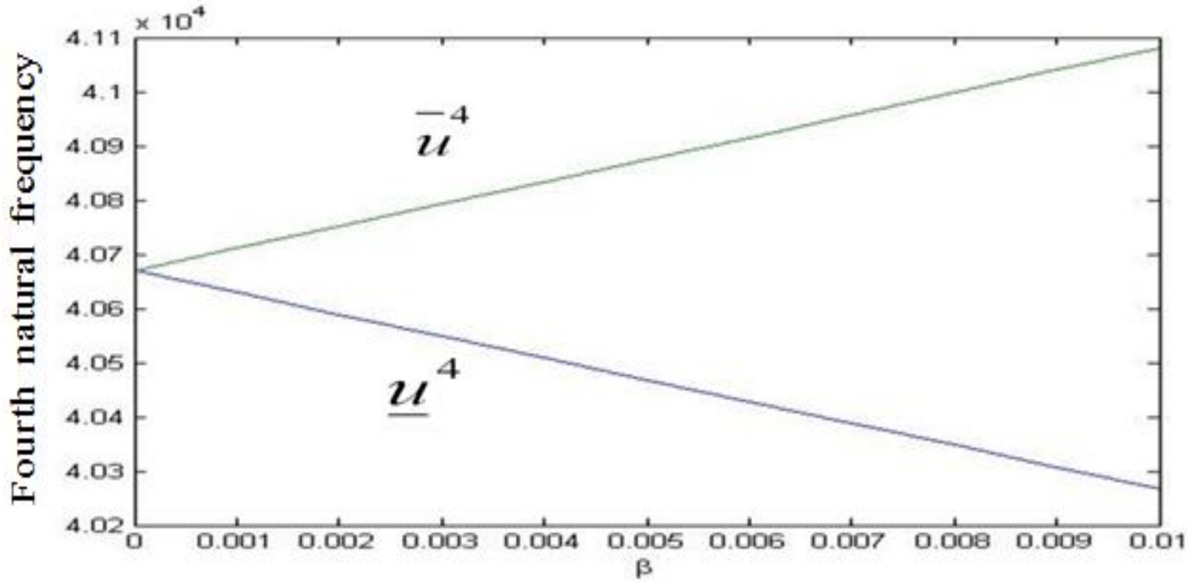

Fig.12 Plot of upper and lower bounds of fourth natural frequency verses the uncertainty factor β for ρ as interval for homogeneous beam having 2 elements.

Table 7 Interval static responses of a beam having 2 elements with uncertain factor $\beta = 1\%$

u	$\underline{u}^i$	$\overline{u}^i$
u^1	10.471625	10.683173
u^2	417.86742	426.30918
u^3	4780.0517	4876.6184
u^4	40267.579	41081.065

5.3 Homogenous beam with density (ρ) and Young's modulus (E) both as interval

In this case, the same beam with both density and Young's modulus as interval is considered. The governing equations satisfying the boundary conditions are obtained again where ρ and E are considered as interval i.e. $\rho = [\underline{\rho}, \overline{\rho}]$ and $E = [\underline{E}, \overline{E}]$. One and two elements equations are incorporated here.

Equation for one element:

$$\frac{\underline{EI}}{l^3}\begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix}\{W\} = \underline{\omega}^2 \frac{\underline{\rho}Al}{420}\begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix}\{W\}, \tag{16}$$

$$\frac{\underline{EI}}{l^3}\begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix}\{W\} = \underline{\omega}^2 \frac{\overline{\rho}Al}{420}\begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix}\{W\}, \tag{17}$$

$$\frac{\overline{EI}}{l^3}\begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix}\{W\} = \overline{\omega}^2 \frac{\overline{\rho}Al}{420}\begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix}\{W\} \text{ and} \tag{18}$$

$$\frac{\overline{EI}}{l^3}\begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix}\{W\} = \underline{\omega}^2 \frac{\underline{\rho}Al}{420}\begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix}\{W\} \tag{19}$$

Equation for two elements:

$$\frac{\underline{EI}}{l^3}\begin{bmatrix} 24 & 0 & -12 & 6l \\ 0 & 8l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\{W\} = \underline{\omega}^2 \frac{\underline{\rho} Al}{420}\begin{bmatrix} 312 & 0 & 54 & -13l \\ 0 & 8l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}\{W\}, \tag{20}$$

$$\frac{\underline{EI}}{l^3}\begin{bmatrix} 24 & 0 & -12 & 6l \\ 0 & 8l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\{W\} = \overline{\omega}^2 \frac{\overline{\rho} Al}{420}\begin{bmatrix} 312 & 0 & 54 & -13l \\ 0 & 8l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}\{W\}, \tag{21}$$

$$\frac{\overline{EI}}{l^3}\begin{bmatrix} 24 & 0 & -12 & 6l \\ 0 & 8l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\{W\} = \underline{\omega}^2 \frac{\overline{\rho} Al}{420}\begin{bmatrix} 312 & 0 & 54 & -13l \\ 0 & 8l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}\{W\}, \text{ and} \tag{22}$$

$$\frac{\overline{EI}}{l^3}\begin{bmatrix} 24 & 0 & -12 & 6l \\ 0 & 8l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\{W\} = \overline{\omega}^2 \frac{\underline{\rho} Al}{420}\begin{bmatrix} 312 & 0 & 54 & -13l \\ 0 & 8l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}\{W\}, \tag{23}$$

Let us take the density and Young's modulus as interval.

$\rho = [7795, 7805] Kg/m^3$, $E = [1.998\times 10^{11}, 2.002\times 10^{11}] N/m^2$, $A = 30 cm^2$, $I = 100 mm^4$, $l = 1m$.

Using these parameters along with Eq. (16) to (23) the obtained natural frequencies are given in table 8.

Table 8 Interval values of frequencies with ρ and E as interval

	No of elements		
	1		2
modes	1	[10.64933757,10.68434667]	[10.54926481,10.6089706]
	2	[1033.788843,1037.187366]	[420.9400814,423.351985]
	3		[4820.337846,4834.90721]
	4		[40442.96979,41046.36896]

Taking ρ and E in terms of β i.e. $\rho = [7800 - 7800 * \beta, 7800 + 7800 * \beta]$ and $E = [2\times10^{11} - \beta\times2\times10^{11}, 2\times10^{11} + \beta\times2\times10^{11}]$ and all other parameters are same. Where β varies from 0 to .01. Using these parameters interval eigenvalues are obtained and the results obtained are depicted in term of plots which is given in fig.13 and fig.14 for 1 element discretization and in fig.15 to 18 for 2 element discretization.

For 1 element discretization:

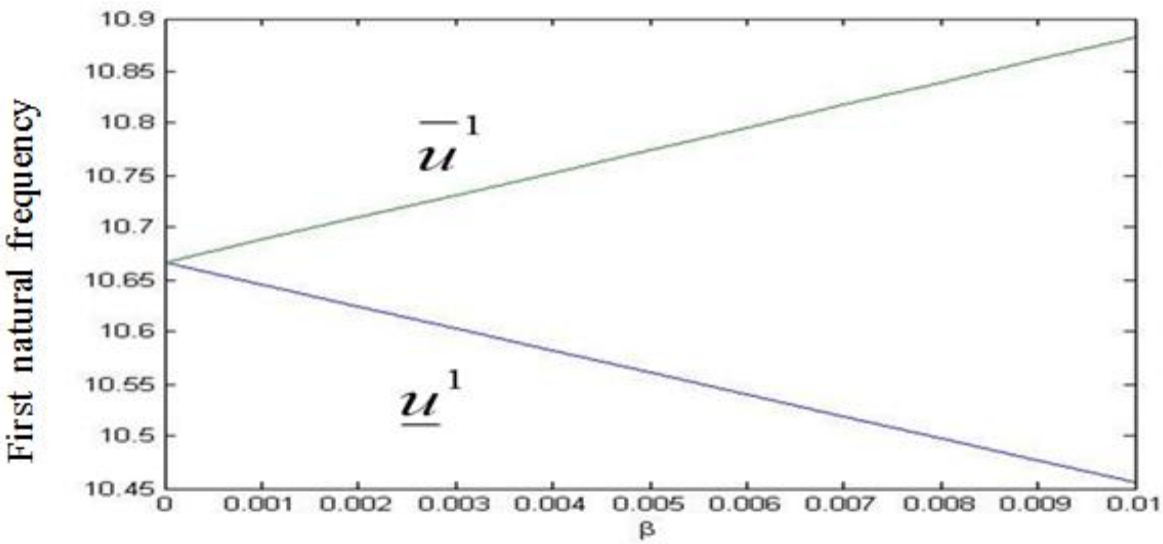

Fig.13 Plot of upper and lower bounds of first natural frequency verses the uncertainty factor β for ρ and E as interval for homogeneous beam having 1 element.

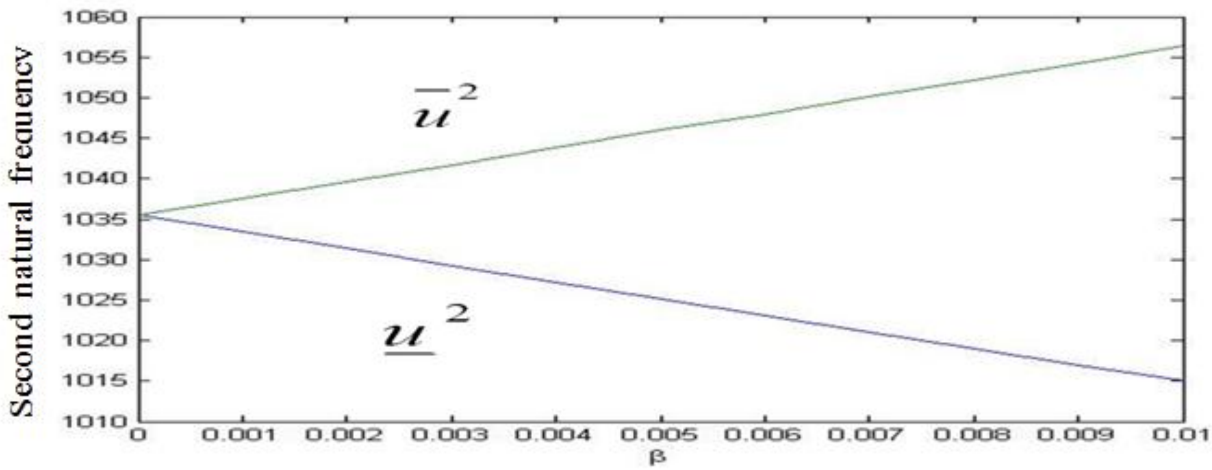

Fig.14 Plot of upper and lower bounds of second natural frequency verses the uncertainty factor β for ρ and E as interval for homogeneous beam having 1 element.

Table 9 Interval static responses of a beam having 1 element with uncertain factor $\beta = 1\%$

u	$\underline{u}^i$	$\overline{u}^i$
u^1	10.455607	10.882322
u^2	1014.9823	1056.4059

For 2 elements discretization:

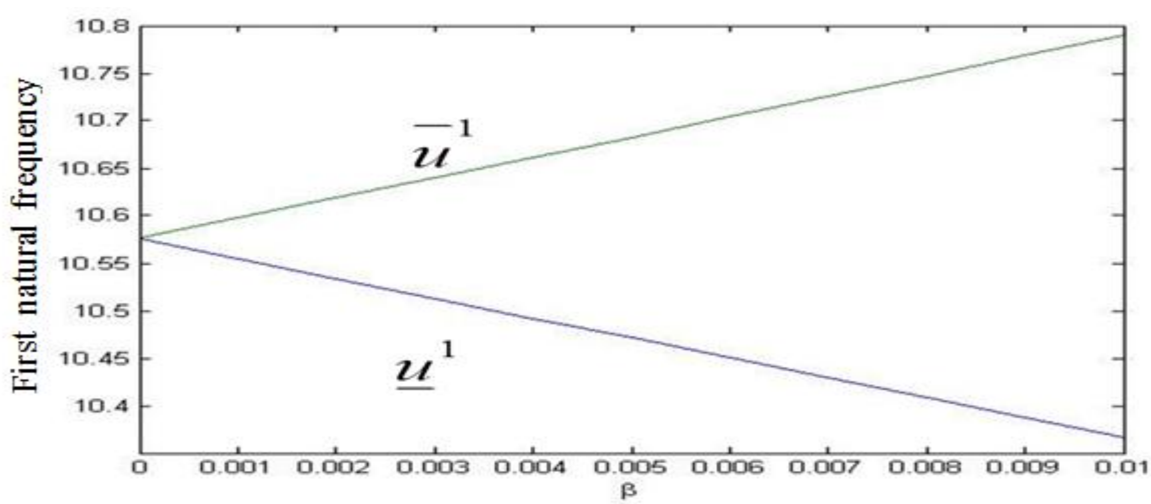

Fig.15 Plot of upper and lower bounds of first natural frequency verses the uncertainty factor β for ρ and E as interval for homogeneous beam having 2 elements.

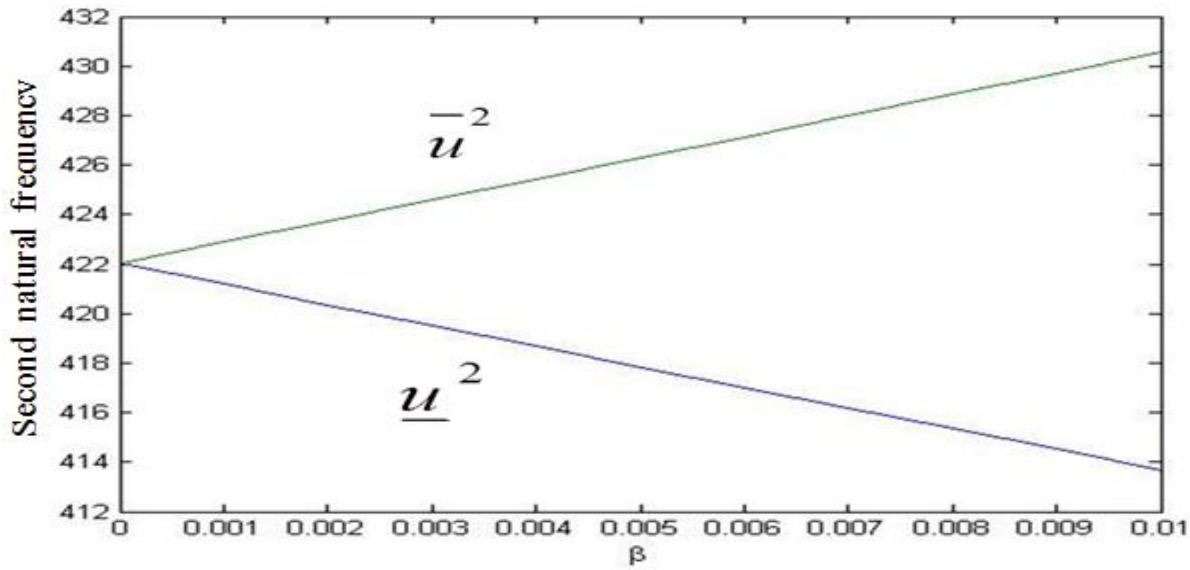

Fig.16 Plot of upper and lower bounds of second natural frequency verses the uncertainty factor β for ρ and E as interval for homogeneous beam having 2 elements.

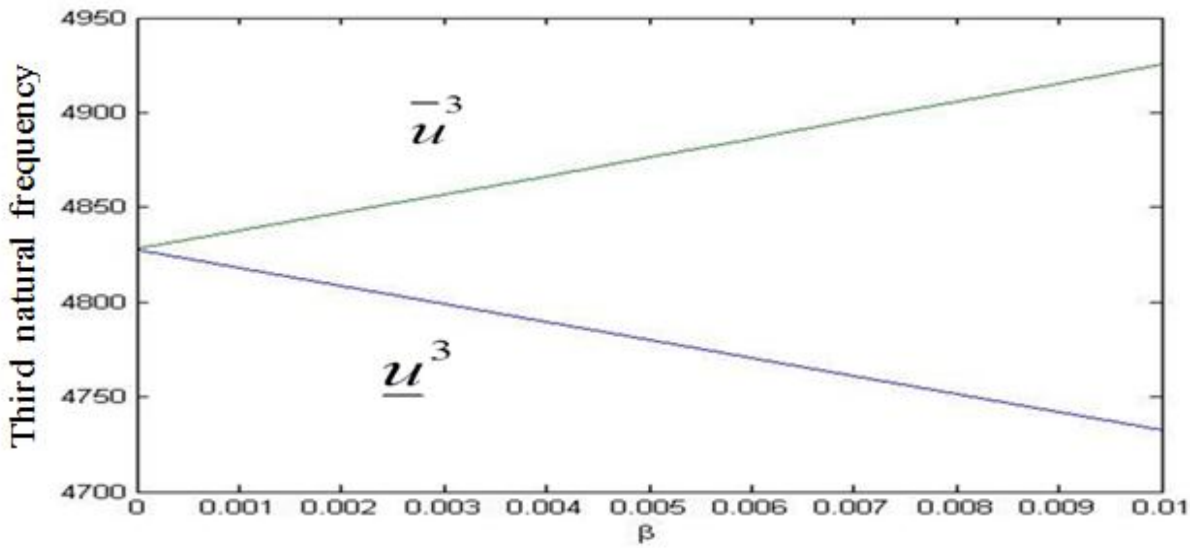

Fig.17 Plot of upper and lower bounds of third natural frequency verses the uncertainty factor β for ρ and E as interval for homogeneous beam having 2 elements.

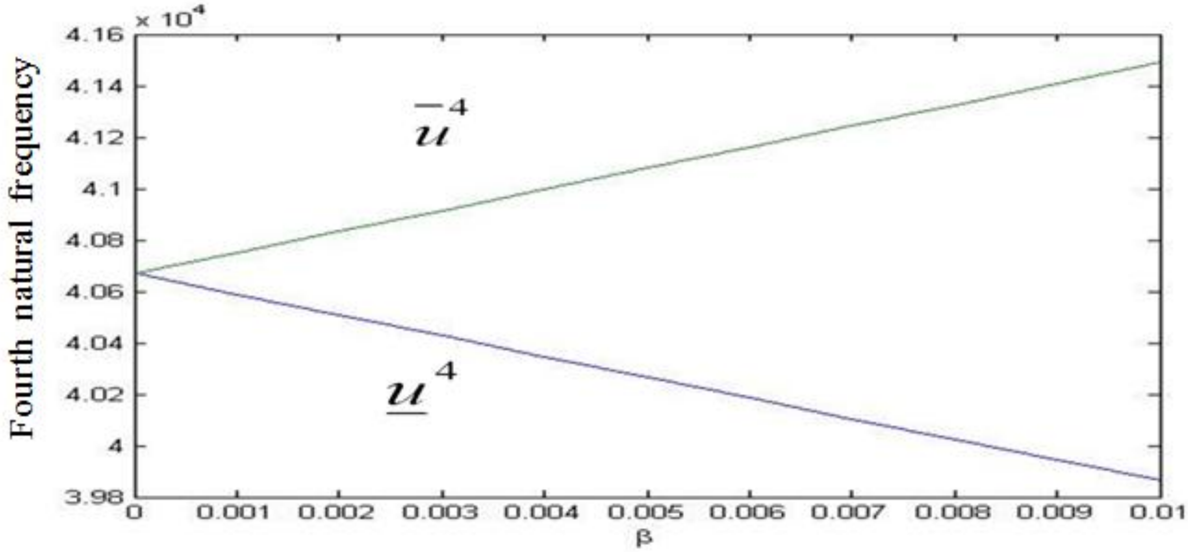

Fig.18 Plot of upper and lower bounds of fourth natural frequency verses the uncertainty factor β for ρ and E as interval for homogeneous beam having 2 elements.

Table 10 Interval static responses of a beam having 2 elements with uncertain factor $\beta = 1\%$

u	$\underline{u}^i$	$\overline{u}^i$
u^1	10.366909	10.790005
u^2	413.68874	430.57228
u^3	4732.2512	4925.3846
u^4	39864.903	41491.876

In all the cases we observe that the crisp value is in-between the interval values.

A non-homogenous beam having crisp material properties is considered. The area of cross-section and moment of inertia varies for different elements along the beam. As such the global mass and stiffness matrices for two elements equations are given below.

Equation for two elements:

$$\frac{E}{l^3}\begin{bmatrix} 12(I_1+I_2) & -6l(I_1-I_2) & -12I_2 & 6lI_2 \\ -6l(I_1-I_2) & 4l^2(I_1+I_2) & -6lI_2 & 2l^2I_2 \\ -12I_2 & -6lI_2 & 12I_2 & -6lI_2 \\ 6lI_2 & 2l^2I_2 & -6lI_2 & 4l^2I_2 \end{bmatrix}\{W\}$$

$$=\frac{\lambda\rho l}{420}\begin{bmatrix} 156(A_1+A_2) & -22l(A_1-A_2) & 54A_2 & -13A_2l \\ -22l(A_1-A_2) & 4l^2(A_1+A_2) & 13A_2l & -3A_2l^2 \\ 54A_2 & 13lA_2 & 156A_2 & -22lA_2 \\ -13lA_2 & -3l^2A_2 & -22lA_2 & 4l^2A_2 \end{bmatrix}\{W\} \qquad (24)$$

Taking the values of the parameters as

$E_i = 2\times10^{11} N/m^2, \rho_i = 7800 Kg/m^3,\ A_1 = 1.44\times10^{-2} m^2,$

$A_2 = 1\times10^{-2} m^2, I_1 = 0.2\times10^{-4} m^4, I_2 = 0.1\times10^{-4} m^4 l = .4m$

(Zhiping et.al [9]

Using these parameters along with Eq. (24), the obtained natural frequencies are given in table 11.

Table.11 Crisp values for natural frequencies for non-homogenous beam

	No of elements	
	2	
modes	1	1445202.724
	2	37142249.62
	3	425266474.1
	4	2950125834

Chapter7. Interval Finite element model for non-homogenous fixed free beam

7.1 Non-homogenous fixed free beam with Moment of inertia as an interval.

Taking $\left[\underline{I}_{(i)}, \overline{I}_{(i)}\right]$, the governing equations for one and two elements according to the same boundary condition are computed. Two element equations are incorporated here.

Equation for two elements:

$$\frac{E}{l^3}\begin{bmatrix} 12(\underline{I}_1+\underline{I}_2) & -6l(\underline{I}_1-\underline{I}_2) & -12\underline{I}_2 & 6l\underline{I}_2 \\ -6l(\underline{I}_1-\underline{I}_2) & 4l^2(\underline{I}_1+\underline{I}_2) & -6l\underline{I}_2 & 2l^2\underline{I}_2 \\ -12\underline{I}_2 & -6l\underline{I}_2 & 12\underline{I}_2 & -6l\underline{I}_2 \\ 6l\underline{I}_2 & 2l^2\underline{I}_2 & -6l\underline{I}_2 & 4l^2\underline{I}_2 \end{bmatrix}\{W\}$$

$$=\frac{\underline{\omega}^2\rho l}{420}\begin{bmatrix} 156(A_1+A_2) & -22l(A_1-A_2) & 54A_2 & -13A_2l \\ -22l(A_1-A_2) & 4l^2(A_1+A_2) & 13A_2l & -3A_2l^2 \\ 54A_2 & 13lA_2 & 156A_2 & -22lA_2 \\ -13lA_2 & -3l^2A_2 & -22lA_2 & 4l^2A_2 \end{bmatrix}\{W\} \text{ and} \qquad (25)$$

$$\frac{E}{l^3}\begin{bmatrix} 12(\overline{I}_1+\overline{I}_2) & -6l(\overline{I}_1-\overline{I}_2) & -12\overline{I}_2 & 6l\overline{I}_2 \\ -6l(\overline{I}_1-\overline{I}_2) & 4l^2(\overline{I}_1+\overline{I}_2) & -6l\overline{I}_2 & 2l^2\overline{I}_2 \\ -12\overline{I}_2 & -6l\overline{I}_2 & 12\overline{I}_2 & -6l\overline{I}_2 \\ 6l\overline{I}_2 & 2l^2\overline{I}_2 & -6l\overline{I}_2 & 4l^2\overline{I}_2 \end{bmatrix}\{W\}$$

$$=\frac{\overline{\omega}^2\rho l}{420}\begin{bmatrix} 156(A_1+A_2) & -22l(A_1-A_2) & 54A_2 & -13A_2l \\ -22l(A_1-A_2) & 4l^2(A_1+A_2) & 13A_2l & -3A_2l^2 \\ 54A_2 & 13lA_2 & 156A_2 & -22lA_2 \\ -13lA_2 & -3l^2A_2 & -22lA_2 & 4l^2A_2 \end{bmatrix}\{W\} \qquad (26)$$

Let us take the values of the parameters as $E_i = 2\times10^{11} N/m^2, \rho_i = 7800 Kg/m^3$,

$A_1 = 1.44\times10^{-2} m^2, A_2 = 1\times10^{-2} m^2, I_1 = \left[0.1998\times10^{-4}, 0.2004\times10^{-4}\right] m^4$,

$I_2 = [0.0999\times10^{-4}, 0.1001\times10^{-4}]m^4$ $l = .4m$

Using these parameters along with Eq. (25) and (26), the obtained natural frequencies are given in table 12.

Table 12 Interval values for natural frequencies for $I_{(i)}$ as interval

	No of elements	
	2	
Modes	1	[1439490.032,1452216.933]
	2	[37110038.97,37190814.96]
	3	[424829565.1,425944949.5]
	4	[2947285802,2953212801]

The obtained natural frequencies are compared with the natural frequencies of non-homogeneous stepped beam having three elements which is obtained with the help of Deif's solution theorem and the parameter vertex solution theorem. The values of the parameters are $\rho_i = 7800Kg/m^3, A_1 = 1.44\times10^{-2}m^2, A_2 = 1\times10^{-2}m^2, A_3 = 0.64\times10^{-2}m^2$, $I_1 = 0.2\times10^{-4}m^4$, $I_2 = 0.1\times10^{-4}m^4$, $I_3 = .05\times10^{-4}m^4 l = .4m, E_1 = [1.997\times10^{11}, 2.003\times10^{11}]N/m^2$, $E_2 = [1.998\times10^{11}, 2.002\times10^{11}]N/m^2, E_3 = [1.999\times10^{11}, 2.001\times10^{11}]N/m^2$

Natural frequencies obtained by Deif's solution theorem are given in Table 13.

Table 13 Interval values for natural frequencies for $E_{(i)}$ as interval

	No of elements	
	3	
Modes	1	[310350.2 , 419675.9]
	2	[7203591 , 7467286]
	3	[48048700 , 48387910]
	4	[257480400 , 258634100]
	5	[890633800 , 893295300]
	6	[2736988000, 2742580000]

Natural frequencies obtained by the parameter vertex solution theorem are given in Table 14.

Table 14 Interval values for natural frequencies for $E_{(i)}$ as interval.

	No of elements	
	3	
Modes	1	[364592.1 , 365569.9]
	2	[7327840 , 7343228]
	3	[48173730 , 48263700]
	4	[257791900 , 258322800]
	5	[891057400 , 892878400]
	6	[2738051000 , 2741501000]

This variation of natural frequencies in Table 12 with Table 13 and Table 14 is because we have taken 2 elements in table 1 whereas 3 elements in Table 13 and Table 14.

Taking I_1 and I_2 in terms of β i.e.

$I_1 = \left[0.2\times10^{-4} - 0.2\times10^{-4}\beta, 0.2\times10^{-4} + 0.2\times10^{-4}\beta\right]$ and
$I_2 = \left[0.1\times10^{-4} - 0.1\times10^{-4}\beta, 0.1\times10^{-4} + 0.1\times10^{-4}\beta\right]$ And all parameters are same. Where β varies from 0 to .01. Using these parameters interval eigenvalues are obtained and the results obtained are depicted in terms of plots which is given in fig.19 to fig.22 for 2 element discretization.

For 2 elements discretization:

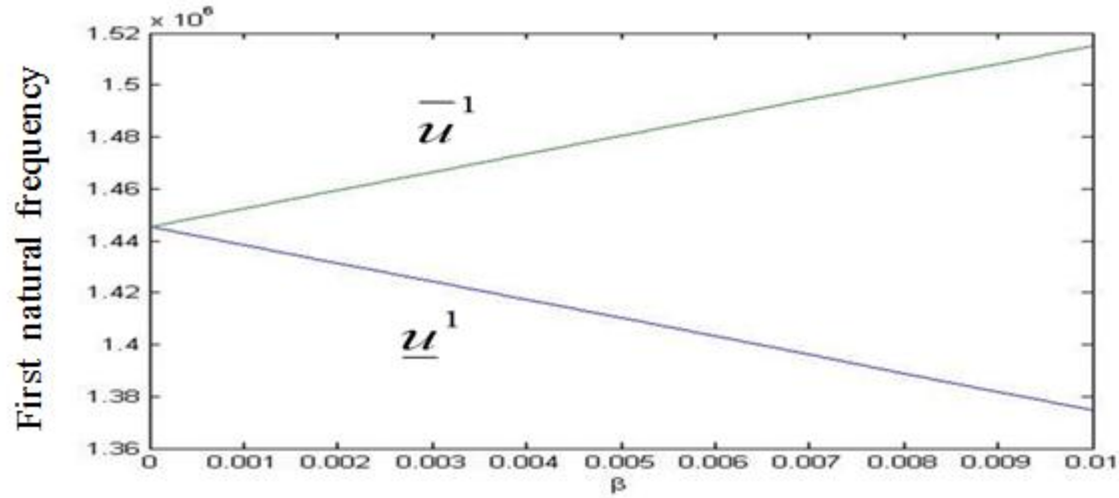

Fig.19 Plot of upper and lower bounds of first natural frequency verses the uncertainty factor β for moment of inertias as interval for non-homogeneous beam having 2 elements.

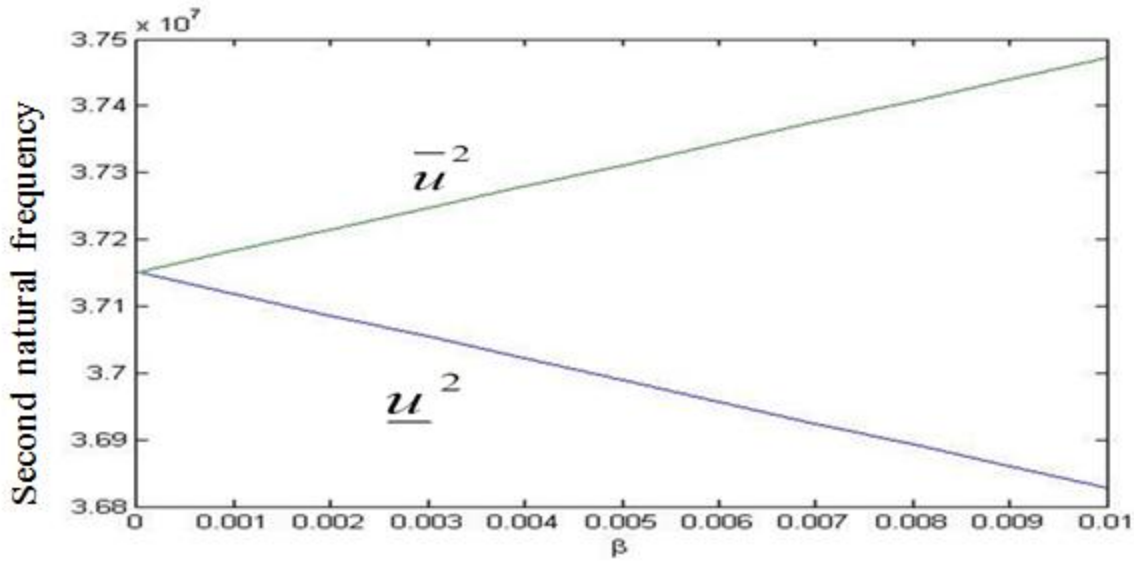

Fig.20 Plot of upper and lower bounds of second natural frequency verses the uncertainty factor β for moment of inertias as interval for non-homogeneous beam having 2 elements.

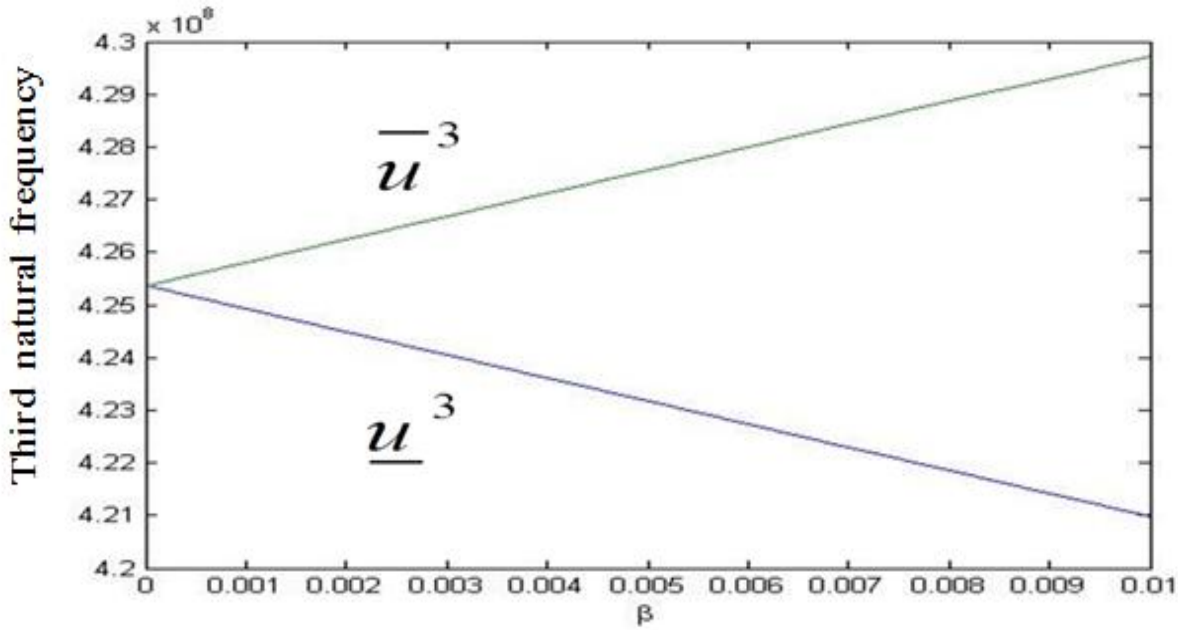

Fig.21 Plot of upper and lower bounds of third natural frequency verses the uncertainty factor β for moment of inertias as interval for non-homogeneous beam having 2 elements.

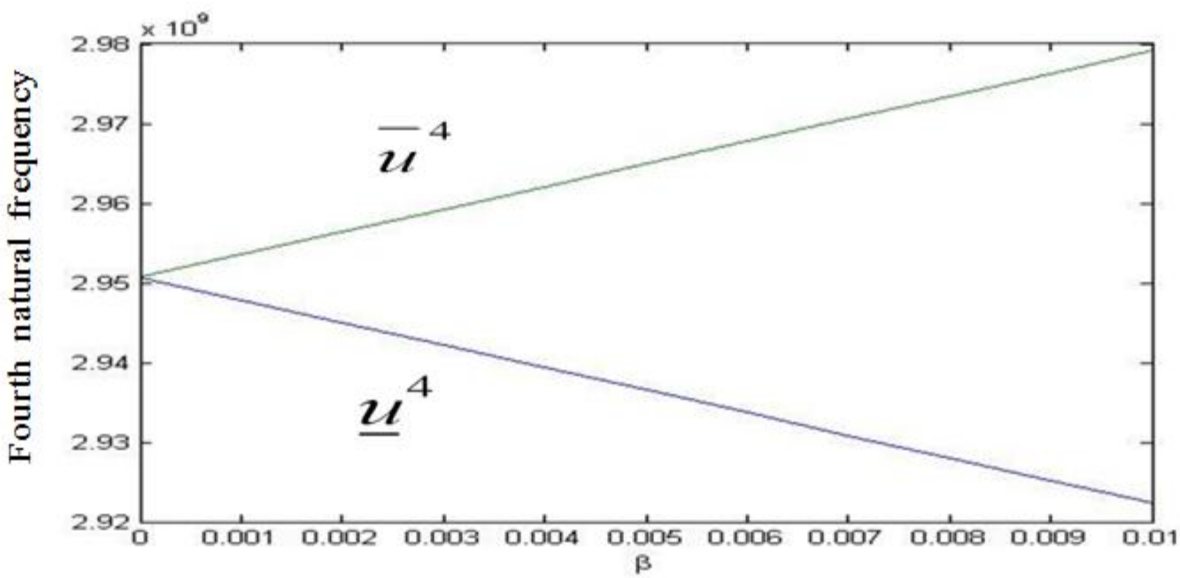

Fig.22 Plot of upper and lower bounds of fourth natural frequency verses the uncertainty factor β for moment of inertias as interval for non-homogeneous beam having 2 elements.

Table 15 Interval static responses of a beam having 2 elements with uncertain factor $\beta = 1\%$

u	$\underline{u}^i$	$\overline{u}^i$
u^1	1374756.372	1515167.845
u^2	36827659.82	37471858.80
u^3	420994057.3	429734512.1
u^4	2922322733.	2979135031.

7.2 Non-homogenous fixed free beam with Area of cross-section as an interval.

A non- homogenous fixed free beam with area of cross-section as interval is considered now. The governing eigenvalue equations satisfying the boundary condition for various elements are obtained. Equations for 2 elements are incorporated here.

Equation for 2 elements:

$$\frac{E}{l^3}\begin{bmatrix} 12(I_1+I_2) & -6l(I_1-I_2) & -12I_2 & 6lI_2 \\ -6l(I_1-I_2) & 4l^2(I_1+I_2) & -6lI_2 & 2l^2I_2 \\ -12I_2 & -6lI_2 & 12I_2 & -6lI_2 \\ 6lI_2 & 2l^2I_2 & -6lI_2 & 4l^2I_2 \end{bmatrix}\{W\}$$

$$= \frac{\underline{\omega}^2\rho l}{420}\begin{bmatrix} 156(\underline{A}_1+\underline{A}_2) & -22l(\underline{A}_1-\underline{A}_2) & 54\underline{A}_2 & -13\underline{A}_2 l \\ -22l(\underline{A}_1-\underline{A}_2) & 4l^2(\underline{A}_1+\underline{A}_2) & 13\underline{A}_2 l & -3\underline{A}_2 l^2 \\ 54\underline{A}_2 & 13l\underline{A}_2 & 156\underline{A}_2 & -22l\underline{A}_2 \\ -13l\underline{A}_2 & -3l^2\underline{A}_2 & -22l\underline{A}_2 & 4l^2\underline{A}_2 \end{bmatrix}\{W\} \text{ and} \qquad (27)$$

$$\frac{E}{l^3}\begin{bmatrix} 12(I_1+I_2) & -6l(I_1-I_2) & -12I_2 & 6lI_2 \\ -6l(I_1-I_2) & 4l^2(I_1+I_2) & -6lI_2 & 2l^2I_2 \\ -12I_2 & -6lI_2 & 12I_2 & -6lI_2 \\ 6lI_2 & 2l^2I_2 & -6lI_2 & 4l^2I_2 \end{bmatrix}\{W\}$$

$$=\frac{\bar{\omega}^2\rho l}{420}\begin{bmatrix} 156(\bar{A}_1+\bar{A}_2) & -22l(\bar{A}_1-\bar{A}_2) & 54\bar{A}_2 & -13\bar{A}_2 l \\ -22l(\bar{A}_1-\bar{A}_2) & 4l^2(\bar{A}_1+\bar{A}_2) & 13\bar{A}_2 l & -3\bar{A}_2 l^2 \\ 54\bar{A}_2 & 13l\bar{A}_2 & 156\bar{A}_2 & -22l\bar{A}_2 \\ -13l\bar{A}_2 & -3l^2\bar{A}_2 & -22l\bar{A}_2 & 4l^2\bar{A}_2 \end{bmatrix}\{W\} \qquad (28)$$

Let us take the values of the parameters as

$E_i = 2\times10^{11} N/m^2, \rho_i = 7800 Kg/m^3,\ A_1 = [1.426\times10^{-2}, 1.454\times10^{-2}]m^2,$

$A_2 = [0.99\times10^{-2}, 1.01\times10^{-2}]m^2,\ I_1 = 0.2\times10^{-4} m^4, I_2 = 0.1\times10^{-4} m^4\ l = .4m$

Using these parameters along with Eq. (27) and (28) the obtained natural frequencies are given in table 16.

Table16 Interval values for natural frequencies for $A_{(i)}$ as interval

	No of elements	
	2	
Modes	1	[1432248.123,1458974.204]
	2	[36802804.21,37504873.48]
	3	[420796198.6,430058834.3]
	4	[294987347,2951992748]

Taking A_1 and A_2 in terms of β i.e.

$A_1 = [1.44\times10^{-2} - 1.44\times10^{-2}\times\beta, 1.44\times10^{-2} + 1.44\times10^{-2}\times\beta]$ and
$A_2 = [1\times10^{-2} - 1\times10^{-2}\times\beta, 1\times10^{-2} + 1\times10^{-2}\times\beta]$ and all parameters are taken same. Where β varies from 0 to .01. Using these parameters interval eigenvalues are obtained and the results obtained are depicted in terms of plots which is given in fig.23 to fig.26 for 2 element discretization.

For 2 elements discretization:

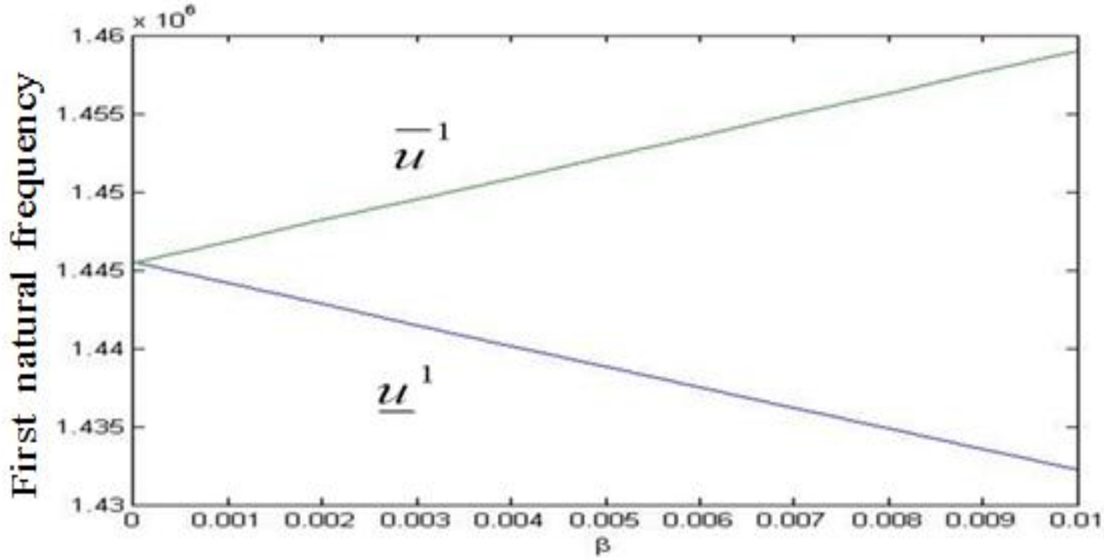

Fig.23 Plot of upper and lower bounds of first natural frequency verses the uncertainty factor β for area of cross-sections as interval for non-homogeneous beam having 2 elements.

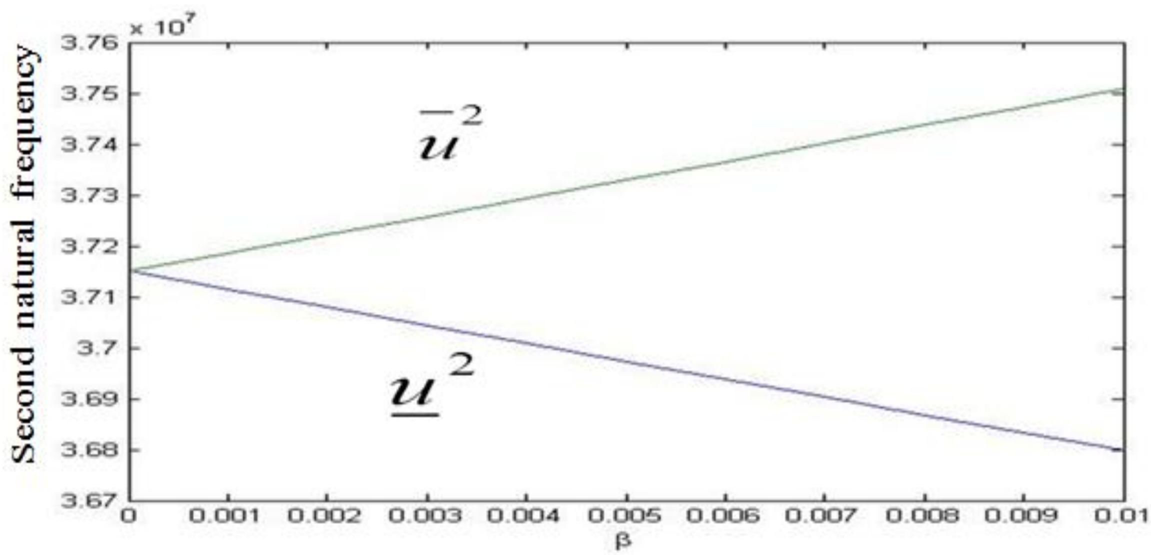

Fig.24 Plot of upper and lower bounds of second natural frequency verses the uncertainty factor β for area of cross-sections as interval for non-homogeneous beam having 2 elements.

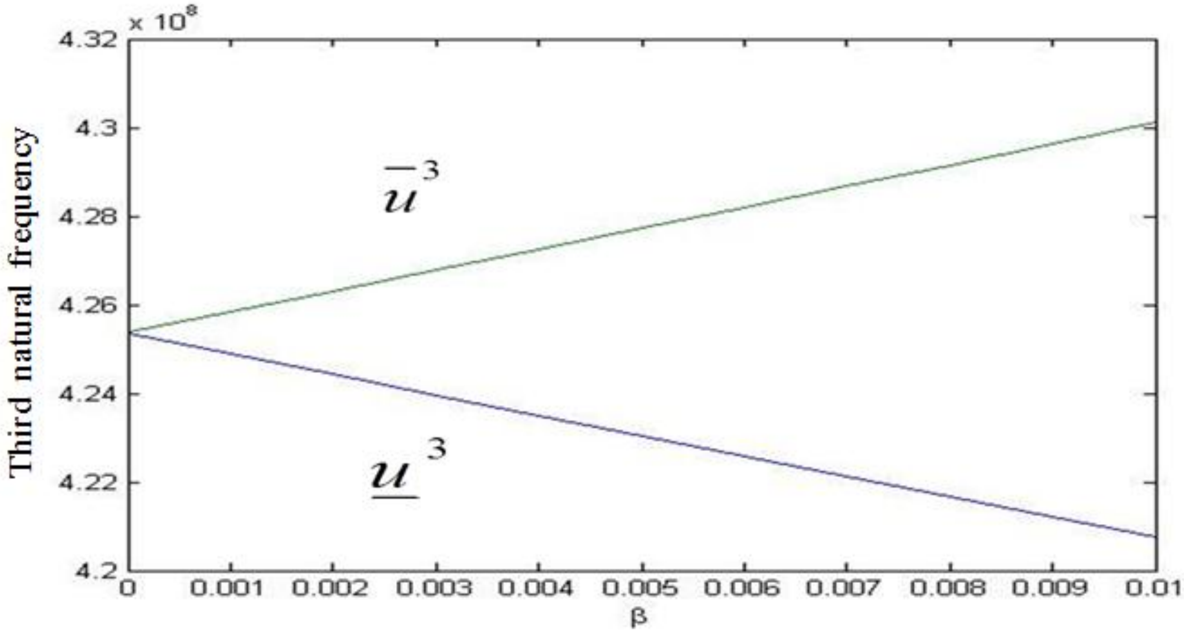

Fig.25 Plot of upper and lower bounds of third natural frequency verses the uncertainty factor β for area of cross-sections as interval for non-homogeneous beam having 2 elements.

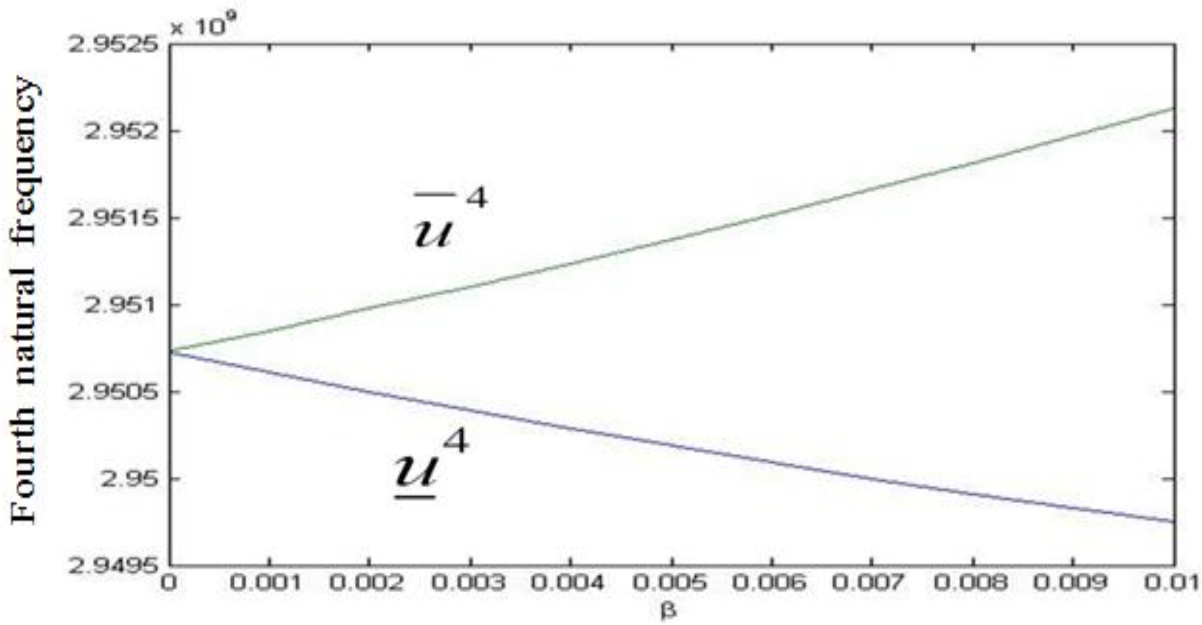

Fig.26 Plot of upper and lower bounds of fourth natural frequency verses the uncertainty factor β for area of cross-sections as interval for non-homogeneous beam having 2 elements.

Table 17 Interval static responses of a beam having 2 elements with uncertain factor $\beta = 1\%$

u	$\underline{u}^i$	$\overline{u}^i$
u^1	1432222.316	1459000.992
u^2	36798382.83	37509451.01
u^3	420742399.0	430116376.4
u^4	2949747850.	2952124382.

7.3 Non-homogenous beam with $I_{(i)}$ and $A_{(i)}$ both as interval.

In this case, the same beam with both area of cross-section and moment of inertia as interval is considered. The governing equations satisfying the boundary condition are obtained again where $I_{(i)}$ and $A_{(i)}$ are considered as interval i.e $I_{(i)} = \left[\underline{I}_{(i)}, \overline{I}_{(i)}\right]$ and $A_{(i)} = \left[\underline{A}_{(i)}, \overline{A}_{(i)}\right]$. Equations for two elements are incorporated here.

Equation for 2 elements:

$$\frac{E}{l^3}\begin{bmatrix} 12(\underline{I}_1+\underline{I}_2) & -6l(\underline{I}_1-\underline{I}_2) & -12\underline{I}_2 & 6l\underline{I}_2 \\ -6l(\underline{I}_1-\underline{I}_2) & 4l^2(\underline{I}_1+\underline{I}_2) & -6l\underline{I}_2 & 2l^2\underline{I}_2 \\ -12\underline{I}_2 & -6l\underline{I}_2 & 12\underline{I}_2 & -6l\underline{I}_2 \\ 6l\underline{I}_2 & 2l^2\underline{I}_2 & -6l\underline{I}_2 & 4l^2\underline{I}_2 \end{bmatrix}\{W\}$$

$$= \frac{\omega^2\rho l}{420}\begin{bmatrix} 156(\underline{A}_1+\underline{A}_2) & -22l(\underline{A}_1-\underline{A}_2) & 54\underline{A}_2 & -13\underline{A}_2 l \\ -22l(\underline{A}_1-\underline{A}_2) & 4l^2(\underline{A}_1+\underline{A}_2) & 13\underline{A}_2 l & -3\underline{A}_2 l^2 \\ 54\underline{A}_2 & 13l\underline{A}_2 & 156\underline{A}_2 & -22l\underline{A}_2 \\ -13l\underline{A}_2 & -3l^2\underline{A}_2 & -22l\underline{A}_2 & 4l^2\underline{A}_2 \end{bmatrix}\{W\}, \quad (29)$$

$$\frac{E}{l^3}\begin{bmatrix} 12(\underline{I}_1+\underline{I}_2) & -6l(\underline{I}_1-\underline{I}_2) & -12\underline{I}_2 & 6l\underline{I}_2 \\ -6l(\underline{I}_1-\underline{I}_2) & 4l^2(\underline{I}_1+\underline{I}_2) & -6l\underline{I}_2 & 2l^2\underline{I}_2 \\ -12\underline{I}_2 & -6l\underline{I}_2 & 12\underline{I}_2 & -6l\underline{I}_2 \\ 6l\underline{I}_2 & 2l^2\underline{I}_2 & -6l\underline{I}_2 & 4l^2\underline{I}_2 \end{bmatrix}\{W\}$$

$$= \frac{\overline{\omega}^2\rho l}{420}\begin{bmatrix} 156(\overline{A}_1+\overline{A}_2) & -22l(\overline{A}_1-\overline{A}_2) & 54\overline{A}_2 & -13\overline{A}_2 l \\ -22l(\overline{A}_1-\overline{A}_2) & 4l^2(\overline{A}_1+\overline{A}_2) & 13\overline{A}_2 l & -3\overline{A}_2 l^2 \\ 54\overline{A}_2 & 13l\overline{A}_2 & 156\overline{A}_2 & -22l\overline{A}_2 \\ -13l\overline{A}_2 & -3l^2\overline{A}_2 & -22l\overline{A}_2 & 4l^2\overline{A}_2 \end{bmatrix}\{W\}, \qquad (30)$$

$$\frac{E}{l^3}\begin{bmatrix} 12(\overline{I}_1+\overline{I}_2) & -6l(\overline{I}_1-\overline{I}_2) & -12\overline{I}_2 & 6l\overline{I}_2 \\ -6l(\overline{I}_1-\overline{I}_2) & 4l^2(\overline{I}_1+\overline{I}_2) & -6l\overline{I}_2 & 2l^2\overline{I}_2 \\ -12\overline{I}_2 & -6l\overline{I}_2 & 12\overline{I}_2 & -6l\overline{I}_2 \\ 6l\overline{I}_2 & 2l^2\overline{I}_2 & -6l\overline{I}_2 & 4l^2\overline{I}_2 \end{bmatrix}\{W\}$$

$$= \frac{\underline{\omega}^2\rho l}{420}\begin{bmatrix} 156(\overline{A}_1+\overline{A}_2) & -22l(\overline{A}_1-\overline{A}_2) & 54\overline{A}_2 & -13\overline{A}_2 l \\ -22l(\overline{A}_1-\overline{A}_2) & 4l^2(\overline{A}_1+\overline{A}_2) & 13\overline{A}_2 l & -3\overline{A}_2 l^2 \\ 54\overline{A}_2 & 13l\overline{A}_2 & 156\overline{A}_2 & -22l\overline{A}_2 \\ -13l\overline{A}_2 & -3l^2\overline{A}_2 & -22l\overline{A}_2 & 4l^2\overline{A}_2 \end{bmatrix}\{W\} \text{ and} \qquad (31)$$

$$\frac{E}{l^3}\begin{bmatrix} 12(\overline{I}_1+\overline{I}_2) & -6l(\overline{I}_1-\overline{I}_2) & -12\overline{I}_2 & 6l\overline{I}_2 \\ -6l(\overline{I}_1-\overline{I}_2) & 4l^2(\overline{I}_1+\overline{I}_2) & -6l\overline{I}_2 & 2l^2\overline{I}_2 \\ -12\overline{I}_2 & -6l\overline{I}_2 & 12\overline{I}_2 & -6l\overline{I}_2 \\ 6l\overline{I}_2 & 2l^2\overline{I}_2 & -6l\overline{I}_2 & 4l^2\overline{I}_2 \end{bmatrix}\{W\}$$

$$= \frac{\overline{\omega}^2\rho l}{420}\begin{bmatrix} 156(\underline{A}_1+\underline{A}_2) & -22l(\underline{A}_1-\underline{A}_2) & 54\underline{A}_2 & -13\underline{A}_2 l \\ -22l(\underline{A}_1-\underline{A}_2) & 4l^2(\underline{A}_1+\underline{A}_2) & 13\underline{A}_2 l & -3\underline{A}_2 l^2 \\ 54\underline{A}_2 & 13l\underline{A}_2 & 156\underline{A}_2 & -22l\underline{A}_2 \\ -13l\underline{A}_2 & -3l^2\underline{A}_2 & -22l\underline{A}_2 & 4l^2\underline{A}_2 \end{bmatrix}\{W\} \qquad (32)$$

Let us take moment of inertias and area of cross-sections both as interval. i.e.

$$I_1 = \left[0.1998\times10^{-4}, 0.2004\times10^{-4}\right]m^4, I_2 = \left[0.0999\times10^{-4}, 0.1001\times10^{-4}\right]m^4$$
$$A_1 = \left[1.426\times10^{-2}, 1.454\times10^{-2}\right]m^2,\ A_2 = \left[0.99\times10^{-2}, 1.01\times10^{-2}\right]m^2,$$
$$E_i = 2\times10^{11} N/m^2, \rho_i = 7800 Kg/m^3,\ l = .4m$$

Using these parameters along with Eq. (29) to (32) the obtained natural frequencies are given in table 18

Table 18 Interval values for natural frequencies for $A_{(i)}$ and $I_{(i)}$ both as interval.

	No of elements	
	2	
modes	1	[1426587.765,1466056.630]
	2	[36770837.05,37553795.33]
	3	[420364627.3,430746599.2]
	4	[2947041158,2955081996]

Taking A_1, A_2, I_1 and I_2 in terms of β i.e.

$$A_1 = \left[1.44\times10^{-2} - 1.44\times10^{-2}\times\beta, 1.44\times10^{-2} + 1.44\times10^{-2}\times\beta\right],$$
$$A_2 = \left[1\times10^{-2} - 1\times10^{-2}\times\beta, 1\times10^{-2} + 1\times10^{-2}\times\beta\right]$$
$$I_1 = \left[0.2\times10^{-4} - 0.2\times10^{-4}\beta, 0.2\times10^{-4} + 0.2\times10^{-4}\beta\right] \text{and}$$

$I_2 = \left[0.1\times10^{-4} - 0.1\times10^{-4}\beta, 0.1\times10^{-4} + 0.1\times10^{-4}\beta\right]$ And all parameters are taken same. Where β varies from 0 to .01. Using these parameters interval eigenvalues are obtained and the results obtained are depicted in terms of plots which is given in fig.27 to fig.30 for 2 element discretization.

For 2 elements discretization:

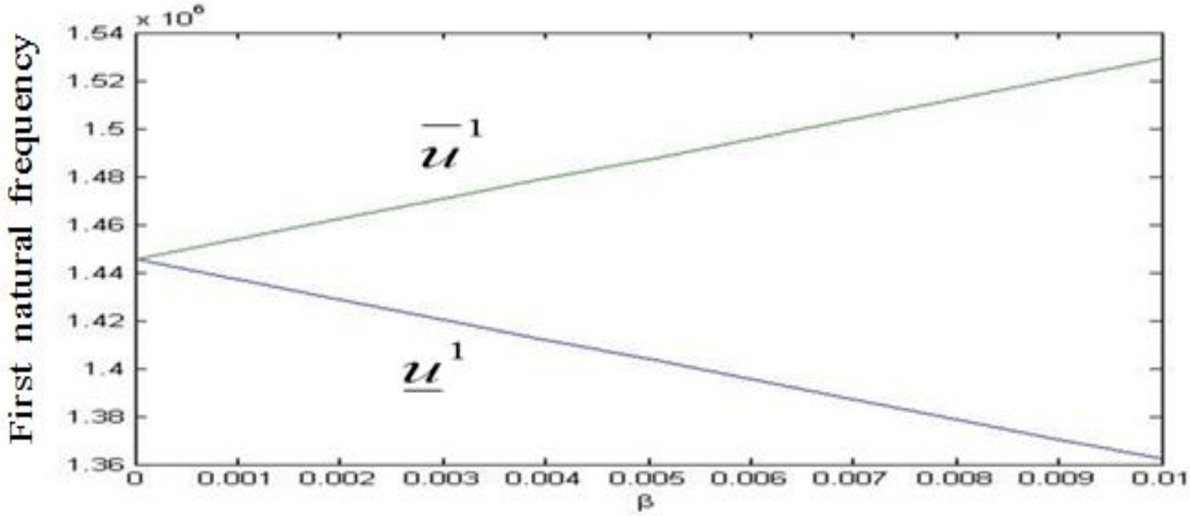

Fig.27 Plot of upper and lower bounds of first natural frequency verses the uncertainty factor β for area of cross-sections and moment of inertias as interval for non-homogeneous beam having 2 elements.

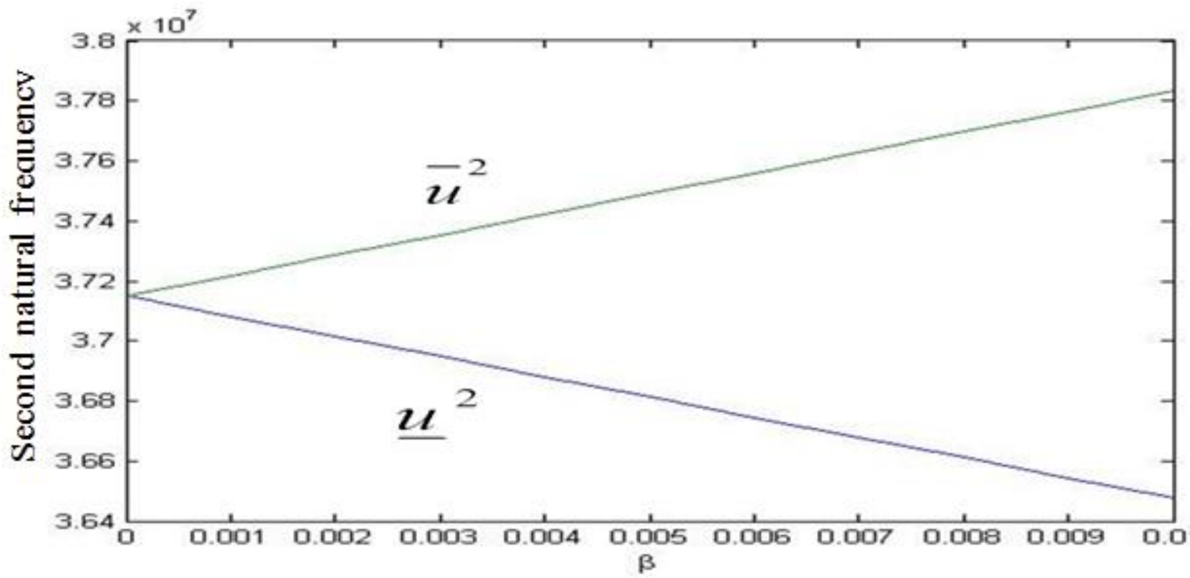

Fig.28 Plot of upper and lower bounds of second natural frequency verses the uncertainty factor β for area of cross-sections and moment of inertias as interval for non-homogeneous beam having 2 elements.

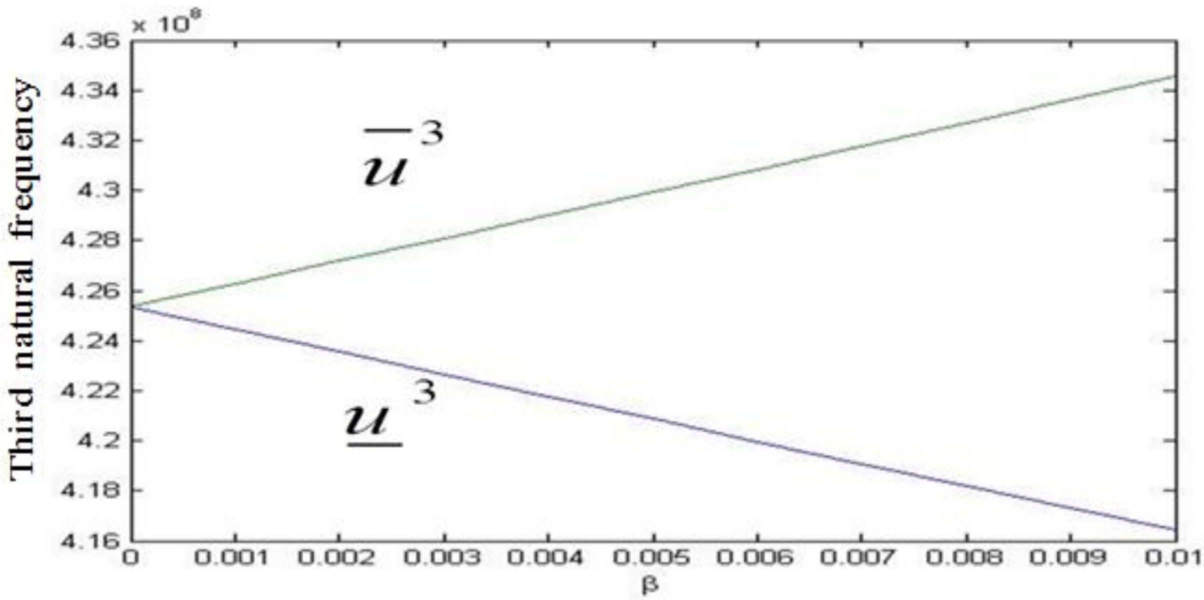

Fig.29 Plot of upper and lower bounds of third natural frequency verses the uncertainty factor β for area of cross-sections and moment of inertias as interval for non-homogeneous beam having 2 elements.

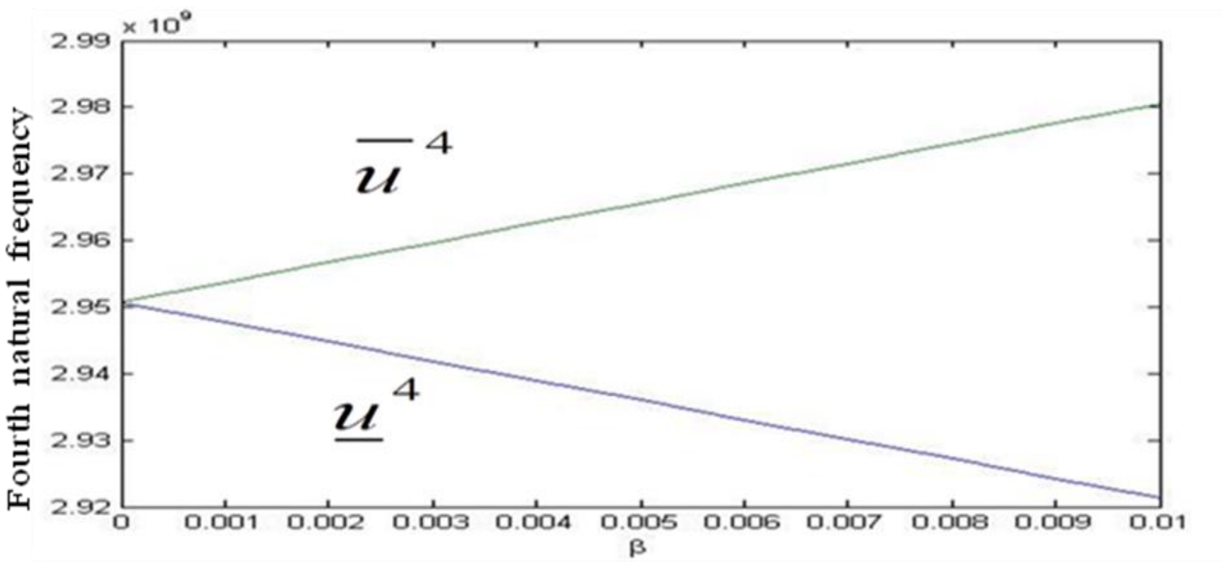

Fig.30 Plot of upper and lower bounds of fourth natural frequency verses the uncertainty factor β for area of cross-sections and moment of inertias as interval for non-homogeneous beam having 2 elements.

Table 19 Interval static responses of a beam having 2 elements with uncertain factor $\beta = 1\%$

u	$\underline{u}^{i}$	$\overline{u}^{i}$
u^1	1374756.372	1515167.845
u^2	36827659.82	37471858.80
u^3	420994057.3	429734512.1
u^4	2922322733.	2979135031.

It may be seen from the above numerical results that the natural frequencies gradually decrease with increase in number of elements as it should be. In crisp values of natural frequency for homogenous beam, the first natural frequency got reduced to 8006.248 from 10.6668309 to 10.57634112. Similar trend of reduction may also be seen for interval cases. Moreover, in Table 2 the interval width for natural frequencies also reduces with increase in elements (first natural frequency reduces to (10.56549416, 10.58664630) from (10.658971, 10.6802363). This is true for only in the density case. However in case of Young's modulus case (as interval) it is almost same. The case of Young's modulus and density both as interval at a time the width again increase as we increase the number of elements. It is interesting to note also that the addition of the computed frequency widths for the cases of homogeneous beam viz. interval (such as Tables 2 and 5) gives the interval width of natural frequencies in Table 8.

Chapter 9. Conclusions and Future Directions

9.1 Conclusions

The investigation presents here the Interval FEM in the vibration of homogeneous and non-homogeneous beam structures. The related generalized eigenvalue problem with respect to the interval components are solved to obtain the natural frequencies depending upon the number of elements taken in the discretization. A method is given to obtain interval eigenvalues. The investigation presented here may find in real application where the material properties may not be obtained in term of crisp values but a vague value in term of uncertain bound is known. The results obtained are depicted in term of plots to show the efficacy of the proposed method.

9.2 Future Directions

The investigation gives a new idea of the Interval FEM through eigenvalue computation and this can very well be used in future research for better results for other eigenvalue problems obtained in different applications. The idea may easily be extended to other structural problems with various complicating effects. Although this require more complex forms of interval computation to handle the corresponding problem.

References:

1. Dimarogonas A.D., Interval Analysis of Vibrating Systems, Journal of Sound and Vibration (1995), 183(4), 739-749;

2. Chen S. H., Lian H.D., Yang X.W., Interval Eigenvalue Analysis for Structures with Interval, Finite Element Analysis and Design (2003), 39, 419-431;

3. Verhaeghe W., Munck M. D., Desmet W., Vandepitte D. and Moens D., A fuzzy finite element analysis technique for structural static analysis based on interval fields, 4th International Workshop on Reliable Engineering Computing (REC 2010), 117-128

4.Gersem H.D., Moens D., Desmet W., Vandepette D. (2004), Interval and fuzzy finite element analysis of mechanical structures with uncertain parameters, proceedings of ISMA,3009-3021

5. N. R. Mahato and S. Chakraverty, Fuzzy Finite Element Method for Vibration Analysis of Imprecisely Defined BAR, Meccanica, 2011 (Communicated).

6. Qui Z., Wang X. and Chen J., Exact bounds for the static response set of structures with uncertain-but-bounded parameters, International journal of solids ad structures (2006), 43, 6574-6593;

7. Jaulin L., Kieffer M., Didrit O. and Walter E.,"Applied Interval Analysis", Springer (2001).

8. R. B. Bhat and S. Chakraverty, Numerical Analysis in Engineering, Alpha Science, (2004);

9. Zhiping Qiu, Xiaojun Wang, Michael I. Friswell, Eigenvalue bounds of structures with uncertain- but-bounded parameters, Elsevier publication. (2003)

10. S. Chakraverty, Vibration of plates, CRC Press, Taylor and Francis Group, (2009);

11. S.S.Rao, The Finite Element Method In Engineering, 4th edition, Elsevier Publication(2005)

12. P.Sesu, Textbook of Finite Element Analysis, PHI learning private limited. (2003)

13.Diptiranjan Behera, S. Chakraverty and D. Datta, Fuzzy Finite Element Approach for Vibration Analysis of Beam with Uncertain Material Properties, Proceedings of 56TH Congress of Indian Society of Theoretical and Applied Mechanics, SVNIT, Surat-395 007, India, 19th - 21st December, 17-24, 2011.

14. Diptiranjan Behera and S. Chakraverty, Vibration Analysis of Imprecisely Defined Multistory Shear Structure, Applied Soft Computing, 2012 (Communicated)

15.Diptiranjan Behera and S. Chakraverty Uncertain Eigenvalues of Imprecisely Defined Structures, Journal of Vibration and Acoustics, 2012 (Communicated)